Lec 9

Review of Poisson Approximation

Further applications of Poisson Process

Recall

Thm. Conditioned on $N(t) = n$, the n arrival time $T_1, \dots, T_n$ has the same distribution as $(V_1, \dots, V_n)$. where $(V_1, \dots, V_n)$ is the ordered version of $(U_1, \dots, U_n)$. and each $U_i \sim \text{Uniform}(0, t)$ independently.

A generalization of thinning.

k types of events. each time s. $P_i(s)$ $\sum_{i=1}^{k} P_i(s) = 1$.

Thm ☆ $N_i(t)$. arrival of type i events.

independent Poisson variables with mean

$$E[N_i(t)] = \lambda \int_0^t P_i(s)\, ds.$$

Example $M/G/\infty$ queue.

Customer arrival $\sim$ Poisson(λ).

infinite number of servers

Service time: independent G.

type 1

$X(t)$. # of customers completed services before t.

$Y(t)$: # of customers being served at time t.

type 2

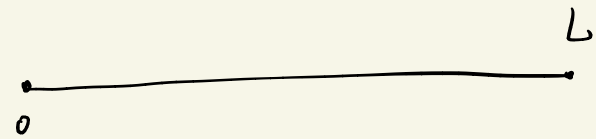
By ☆. $P_X(s) = G(t-s)$. $P_Y(s) = 1 - G(t-s) =: \overline{G}(t-s)$

$$E[X(t)] = \lambda \int_0^t P_X(s)\, ds = \lambda \int_0^t G(t-s)\, ds = \lambda \int_0^t G(y)\, dy$$

$$E[Y(t)] = \lambda \int_0^t P_Y(s)\, ds = \lambda \int_0^t \overline{G}(t-s)\, ds = \lambda \int_0^t \overline{G}(y)\, dy.$$

Example. # of encounters.

highway



cars arrives $\sim$ Poisson(λ).

Constant speed $\sim G$.

What is the speed minimizing encounters?

Choose speed x Travel time $t_0 = L/x$

$\Rightarrow$ Enter at time s, leave at time $s + t_0$.

Distribution of the travel time.

$$F(t) = P_r[T \le t] = P[L/x \le t]$$

$$= \bar{G}(L/t).$$

Type 1 car: encounter with you:

① enter earlier than s, leave later than $s+t_0$.

② enter later than s, leave earlier than $s+t_0$.

① $t<s \Rightarrow t+T > s+t_0$

② $t>s \Rightarrow t+T < s+t_0$.

$$p_1(t) = \begin{cases} P(t+T > t+t_0) = \bar{F}(s+t_0-t) & \text{if } t<s. \\ P(t+T < s+t_0) = F(s+t_0-t) & \text{if } s<t<s+t_0. \\ 0, & \text{if } t>s+t_0. \end{cases}$$

$$E[\text{type 1 cars}] = \lambda \int_0^\infty p_1(t)\, dt$$

$$= \lambda \left(\int_0^s \bar{F}(s+t_0-t)\, dt + \int_s^{s+t_0} F(s+t_0-t)\, dt \right)$$

$$= \lambda \int_{t_0}^{s+t_0} \bar{F}(y)\, dy + \lambda \int_0^{t_0} F(y)\, dy. \quad \cdots (\heartsuit).$$

$$\frac{d\heartsuit}{dt_0} = \lambda \left(\bar{F}(s+t_0) - \bar{F}(t_0) + F(t_0) \right) = 0$$

$\bar{F}(s+t_0) \approx 0$ for large s.

$\Rightarrow \quad F(t_0) = \overline{F(t_0)}$

$\Rightarrow \quad F(t_0) = \frac{1}{2}$.

The optimal speed is the median of G.

<u>Example</u>. Tracking the # of HIV infections.

# of Individuals contract HIV $\sim$ Poi(λ). — unknown

incubation time $\sim G$ — known.

type 1. symptoms shown.

type 2. symptoms not shown.

$$E[N_1(\lambda)] = \lambda \int_0^t G(t-s)\, ds = \lambda \int_0^t G(y)\, dy$$

$$E[N_2(\lambda)] = \lambda \int_0^t \bar{G}(y)\, dy.$$

Let $\hat{n}_1 = E[N_1(\lambda)]$. $\quad \hat{\lambda} = \hat{n}_1 \Big/ \int_0^t G(y)\, dy$

$$\Rightarrow \hat{n}_2 = \hat{n}_1 \cdot \frac{\int_0^t \bar{G}(y)\, dy}{\int_0^t G(y)\, dy}.$$

$\star$. Each $N_i(t)$ is a non-homogeneous Poisson Process

with $\lambda(t) = \lambda P_i(t)$.

Nonhomogeneous Poisson Processes

$\{N(t): t \geq 0\}$ rate $\lambda(s)$. if

① $N(0) = 0$

② $N(t)$ has independent increment.

③. $N(t) - N(s)$ is Poisson with mean $\int_s^t \lambda(r)\, dr$.

proof of $\star$.

$$P_r[N_1(t) = n_1, \dots, N_k(t) = n_k]$$

$$= P_r\left[N_1(t) = n_1, \dots, N_k(t) = n_k \,\middle|\, N(t) = \sum_{i=1}^{k} n_i\right] \cdot P_r\left[N(t) = \sum_{t=1}^{k} n_i\right].$$

$$P_i = \frac{1}{t}\int_0^t P_i(s)\, ds.$$

$$= \left(\frac{\left(\sum_{i=1}^{k} n_i\right)!}{n_1! \cdots n_k!} P_1^{n_1} \cdots P_k^{n_k} \right) \cdot e^{-\lambda t} \frac{(\lambda t)^{\sum n_i}}{(\sum n_i)!}$$

$$= \prod_{i=1}^{k} e^{-\lambda t P_i} (\lambda t P_i)^{n_i} / n_i!$$