Lec 7. Poisson Process.

Example. The number of customers a restaurant serves.

Day 1.	100
Day 2.	120
Day 3.	80
Day 4.	75
Day 5	110

What is the distribution?

So one knows how to prepare.

Assumption. Divide a day into n slots for very large n.

In each slot, a customer arrives with Ber(p).

Probability of k customers.

$$\lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \lim_{n\to\infty} \boxed{\binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}} \triangleq F(n).$$

$$F(n) = \frac{n!}{(n-k)!\,k!} \left(\frac{\lambda}{n}\right)^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-k}.$$

$k = o(n)$

$\underline{\underline{n\to\infty}}$. $\underline{\frac{\lambda^k}{k!} e^{-\lambda}}$ Poisson Distribution (approximation of binomial distribution for small p).

In the example. $\lambda = 9]$.

$\Pr[X \le 110] = 0.912642$.

$X \sim \text{Poisson}(\lambda)$.

$$\mathbb{E}X = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = \lambda.$$

# of Customers in t days? $\text{Poisson}(\lambda t)$.

$\underline{\text{Prop}}$. $X_i \sim \text{Poisson}(\lambda_i)$. $\sum X_i \sim \text{Poisson}(\sum \lambda_i)$.

Only verify that $X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$.

$$P(X_1+X_2=n) = \sum_{m=0}^{n} P(X_1=m)\,P(X_2=n-m)$$

$$= \sum_{m=0}^{n} e^{-\lambda_1}\frac{\lambda_1^m}{m!}\cdot e^{-\lambda_2}\frac{\lambda_2^{n-m}}{(n-m)!}$$

$$= e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}$$

$$= e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^n}{n!}$$

Def. $\{N(s),\ s\geq 0\}$ is a Poisson process if

① $N(0)=0$

② $N(t+s)-N(s) = \text{Poisson}(\lambda t)$.

③ $N(t)$ has independent increments :

$\forall\ t_0 < t_1 < \cdots < t_n$

$N(t_1)-N(t_0),\ N(t_2)-N(t_1),\ \ldots,\ N(t_n)-N(t_{n-1})$ mutually independent.

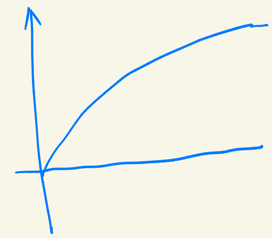
Exponential Distribution

T follows exponential distribution with rate λ.

if $P[T \le t] = 1 - e^{-\lambda t}$. $\forall t \geq 0$.

$\quad \| $

$F(t)$.



density $f(t) = F'(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 . \end{cases}$

Prop. $\tau_1, \tau_2, \ldots$ independent exponential (λ).

$T_n = \tau_1 + \tau_2 + \cdots + \tau_n$. $T_0 = 0$.

$N(s) = \max\{n : T_n \le s\}$. is a Poisson Process.

Another definition of Poisson Process.

Basic Properties of T

$$\mathbb{E}T = \int_0^\infty \lambda t e^{-\lambda t} dt = -\int_0^\infty t \, de^{-\lambda t}$$

$$= e^{-\lambda t} \cdot t \Big|_\infty^0 + \int_0^\infty e^{-\lambda t} dt$$

$$= -\frac{1}{\lambda}\left(e^{-\lambda t}\Big|_0^\infty\right) = \frac{1}{\lambda}.$$

$$\mathbb{E}T^2 = \int_0^\infty \lambda t^2 e^{-\lambda t} dt = -\int_0^\infty t^2 \, de^{-\lambda t}$$

$$= t^2 e^{-\lambda t}\Big|_\infty^0 + 2\int_0^\infty e^{-\lambda t} dt = 2/\lambda^2.$$

$$Var(T) = 1/\lambda^2.$$

Lack of Memory

$$p(T > t+s \mid T > t) = T(T > s).$$

pf. $$p(T > t+s \mid T > t) = \frac{p(T > t+s)}{p(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$= P_r[T > s].$$

Thm. $\tau_1, \dots, \tau_n$ independent $\exp(\lambda)$. The sum $T_n = \sum_{i=1}^{n} \tau_i$ follows $\Gamma(n, \lambda n)$, namely

$$f_{T_n}(t) = \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{for } t \geq 0.$$

Pf. Induction on n.

$n=1$. $f_{T_1}(t) = \lambda e^{-\lambda t}$.

$$f_{T_{n+1}}(t) = \int_0^t f_{T_n}(s)\, f_{\tau_{n+1}}(t-s)\, ds$$

$$= \int_0^t \lambda e^{-s} \frac{(\lambda s)^{n-1}}{(n-1)!} \cdot \lambda e^{-\lambda(t-s)}\, ds$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda t} \int_0^t s^{n-1}\, ds$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda t} \frac{t^n}{n} = \lambda e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Back to Poisson

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mutually independent.

<u>Lem</u>. $N(s) \sim \text{Poisson}(\lambda s)$.

pf. $N(s) = n \iff T_n \le s < T_{n+1}$

$$\Pr[N(s)=n] = \int_0^s f_{T_n}(s) \cdot P(\tau_{n+1} > s-t)\, dt$$

$$= \int_0^s \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda(s-t)}\, dt$$

$$= \frac{\lambda^n}{(n-1)!} e^{-\lambda s} \int_0^s t^{n-1}\, dt = e^{-\lambda s} \frac{(\lambda s)^n}{n!}$$

② ③. then follows from lack of memory property of $\text{Exp}(\lambda)$.

<u>Ex</u>. What is the expected time until the tenth Customer?

$10/\lambda$.

What is the probability that the time between 10th and 11th > 0.1

$$\Pr[\tau_{11} > 0.1] = e^{-0.1\lambda}$$