Lec 6

Reversible Chains

$$\pi(x)\,P(x,y) = \pi(y)\,P(y,x).$$

Detailed Balance Condition.

Metropolis Algorithm

Simulated Annealing 模拟退火

Problem: $S = \{0,1\}^n$. each $x \in S$ has weight $w(x)$.

Output $x^* = \arg\min_{x\in S} w(s)$. $S^* = \{x^* : w(x^*) = \min_x w(x)\}$

The transition graph $G = (S, E)$.

Example

$\forall T > 0, \quad \mu_T(i) \sim e^{-w(i)/T}. \quad \mu^* = \mathrm{Unif}(S^*).$

<u>Prop</u>. $\forall x: \lim_{T\to 0} \mu_T(x) = \mu^*(x)$

Pf. If $w(x) > w(y)$: $e^{-w(x)/T} \Big/ e^{-w(y)/T}$

$$= e^{\frac{w(y)-w(x)}{T}} \xrightarrow{T\to 0} 0$$

<u>A cooling schedule</u>

$T_1, T_2, \ldots, T_n, \ldots$

Transition. $A_1, \ldots, A_n, \ldots$ A_i: Metropolis Rule for μ_{T_i}.

<u>Notations</u>

$$A^{(n)} = \prod_{k=0}^{n-1} A_{T_k}.$$

S_c : non local maxima.

$$r \triangleq \min_{x\in S_c} \max_{y\in S} d(x,y).$$

$$L \triangleq \max_{x\in S} \max_{y\in N(x)} |c(x)-c(y)|.$$

Thm. If a cooling schedule $T_1, T_2, \dots$ satisfies

① $T_i \downarrow 0$.

② $\sum_i \exp\left({-rL}/{T_{ir-1}} \right) = \infty$.

then $\| A^{(n)}(x, \cdot) - \mu^* \| \xrightarrow{n \to \infty} 0$ for all $x \in S$

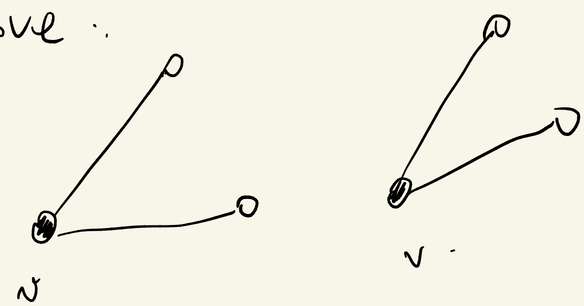
Ex. $T_n = \gamma / \log n$ for $\gamma > rL$.

Sampling Proper Colorings

Metropolis.

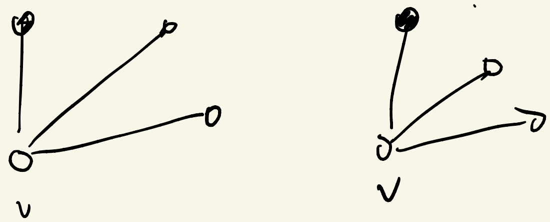
1. pick $v \in V$, $c \in [q]$ u.a.r.
2. Color v with c if possible.

Good move:



$\geq \frac{1}{n} \cdot \frac{q - 2\Delta}{q}$

Bad move



$$\le \frac{d(X_t, Y_t) \cdot \Delta \cdot 2}{n \cdot q}$$

$$\mathbb{E}\left[d(X_{t+1}, Y_{t+1}) \mid (X_t, Y_t) \right] \le d(X_t, Y_t) + \frac{\Delta\, d(X_t, Y_t)}{n} \cdot \frac{2}{q}$$

$$- \frac{d(X_t, Y_t)}{n} \cdot \frac{q - 2\Delta}{q}$$

$$\le d(X_t, Y_t) \left(1 - \frac{q - 4\Delta}{nq}\right).$$