Lec5

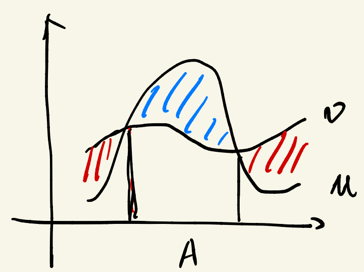
There exists a monotone coupling between $\{X_t\}$, $\boxed{\{Y_t\}}$ random walk.

$\forall t \quad X_t \geq Y_t$. remaining left as exercise.

More on Coupling

total variation distance between two distribution. μ, ν.

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\mu(x) - \nu(x)| = \max_{A \subseteq \Omega} \mu(A) - \nu(A).$$



Coupling Lemma

C a coupling of μ, ν.

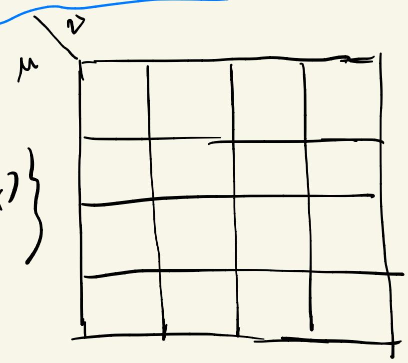
$\Pr_{(X,Y) \sim C}[X \neq Y] \geq \|\mu - \nu\|_{TV}$. $\exists\, C^*$ achieves the equality.

Pf. $\forall A \subseteq \Omega$. C

$$\mu(A) - \nu(A) = \Pr_{(X,Y)\sim C}[X \in A] - \Pr_{(X,Y)\sim C}[Y \in A]$$

$$= \Pr[X \in A \wedge Y \in A] + \Pr[X \in A \wedge Y \notin A] - \Pr[Y \in A]$$

$$\le \Pr[X \in A \wedge Y \notin A] \le \Pr[X \neq Y]$$



Or. $\Pr[X = Y] \le \sum_{x \in \Omega} \min\{\mu(x), \nu(x)\}$

$$\Pr[X \neq Y] \geq 1 - \sum_{x} \min\{\mu(x), \nu(x)\}$$

$$= \sum_{x} \mu(x) - \min\{\mu(x), \nu(x)\}$$

$$= \|\mu - \nu\|_{TV}.$$

Recall FTMC.

Irreducibility + Aperiodic + Positive Recurrence

$\Rightarrow$ Unique stationary & Convergence.

Proof.

We already know (PR) + (I) $\Rightarrow$ (U). $\rightsquigarrow$ λ.

Only need to prove (C).

Consider two chains:

$\{Y_t\}$, $Y_0 \sim \lambda$. $X_0 \sim \mu_0$.

Define a coupling: If $X_t = Y_t$ for some t. then let

$X_{t'} = Y_{t'}$ for all $t' > t$.

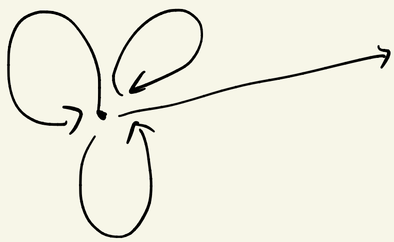
By coupling lemma.

$$\|\underbrace{\mu_t}_{X_t} - \underbrace{\lambda}_{Y_t}\|_{TV} \le \Pr[X_t \neq Y_t]$$

<u>lem.</u> $(I) \Rightarrow \forall i,j. \quad \exists n \quad p^n(i,j) > 0.$

$(I) + (A) \Rightarrow \exists n \quad \forall i,j \quad p^n(i,j) > 0.$

$ax + by + cz = n.$



$$\Pr[X_n = Y_n] \geq \Pr[X_n = Y_n = j] \geq \alpha > 0.$$

$$\Pr[X_n \neq Y_n] \leq 1 - \alpha.$$

$$\Pr[X_{2n} \neq Y_{2n}] = \Pr[X_{2n} \neq Y_{2n} \mid X_n \neq Y_n] \cdot \Pr[X_n \neq Y_n]$$

$$\leq (1-\alpha)^2.$$

Time Reversible.

$\pi(x) P(x,y) = \pi(y) P(y,x).$ Detailed balancing equation.

<u>Metropolis Algorithm.</u>

Random walk $\rightarrow$ Any distribution.