# Lec 4

## Galton - Watson Process

evolution of Y - chromosome / family name.

$G_t$ : # of males at t-th generation.

$X_{ti}$ : # of sons of i-th father in t-th generation.

$G_0 = 1$. $G_{t+1} = \sum_{i=1}^{G_t} X_{ti}$.

$X_{ti}$ : iid with $\forall k \in \mathbb{N}$. $\Pr[X_{ti} = k] = f(k)$.

$\{G_t\}$ - Markov chain.

Assume $f(0) > 0$, $f(0) + f(1) < 1$.

$p \triangleq \Pr[\text{extinction}] = \Pr[G_t = 0 \text{ for some finite } t]$.

$$p = \sum_{k=0}^{\infty} \Pr[G_1 = k] \cdot \Pr[\text{extinction} \mid G_1 = k]$$

$$= \sum_{k=0}^{\infty} f(k) \cdot p^k \triangleq \psi(p).$$

$\psi(z) = \sum f(k) z^k$ is the probability generation function

$p = \psi(p)$ is a fix point

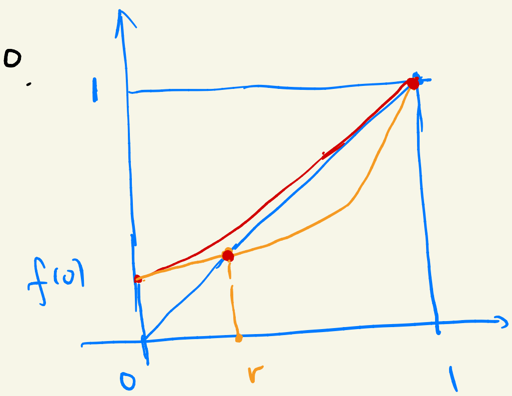
# Property of $\psi$: on $[0,1]$

* $\psi' = \sum_{k=1}^{\infty} k f(k) \cdot z^{k-1} \geq 0.$

* $\psi(1) = 1.$ $\psi(0) = f(0) > 0.$

* $\psi'' = \sum_{k=2}^{\infty} k(k-1) f(k) z^{k-2}$

  $\geq 0.$




which fix point is $\rho$ ?

$\psi$ convex

Case 1: $\psi'(1) = \sum_{k=1}^{\infty} k f(k) = E[Y] \leq 1.$

$\rho = 1.$

Case 2: $E[Y] > 1.$ $\rho = 1$ or $\rho = r$ ?

$P_t \triangleq P_r[G_t = 0]$

Then $P_t \to \rho.$

Now prove $P_t \leq r.$

Induction on $t$:

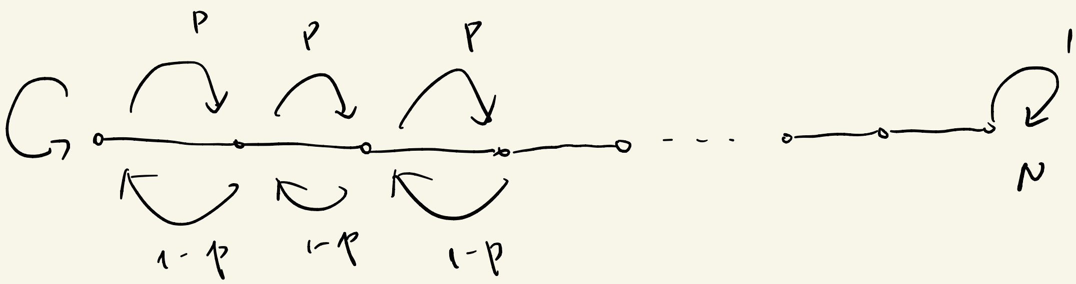
$$P_1 = f(0) < 1.$$

$$P_{t+1} = \Pr[G_{t+1} = 0] = \sum_{k=0}^{\infty} P[G_1 = k] \cdot \Pr[G_{t+1} = 0 \mid G_1 = k]$$

$$= \sum_{k=0}^{\infty} f(k) \cdot P_t^{k} = \psi(P_t).$$

$$\le \psi(r) = r.$$

## Gambler's Ruin




$X_0 = i$. What is the prob. of ending at $N$?

$P_0 = 0$. $P_N = 1$.

$\forall i > 0$: $\quad P_i = (1-p) P_{i-1} + p P_{i+1}$

$$\Rightarrow P_{i+1} = \frac{1}{p} \cdot P_i - \frac{1-p}{p} P_{i-1}.$$

second-order recurrence.

$$\begin{bmatrix} P_{i+1} \\ P_i \end{bmatrix} = \begin{bmatrix} \frac{1}{p} & \frac{p-1}{p} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_i \\ P_{i-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{p} & \frac{p-1}{p} \\ 1 & 0 \end{bmatrix}^{i} \begin{bmatrix} P_1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{p} & \frac{p-1}{p} \\ 1 & 0 \end{bmatrix}. \quad |A - \lambda I| = \begin{vmatrix} \frac{1}{p} - \lambda & \frac{p-1}{p} \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - \frac{1}{p}\lambda + \frac{1-p}{p} = 0$$

$$\Leftrightarrow p\lambda^2 - \lambda + (1-p) = 0$$

$$(\lambda - 1)(p\lambda + p - 1) = 0$$

$$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = \frac{1-p}{p}.$$

If $p \neq \frac{1}{2}$ $\quad A^i = \Lambda \begin{bmatrix} 1 & \\ & (\frac{1-p}{p})^i \end{bmatrix} \Lambda^{-1}.$

$$\Rightarrow P_i = a + b \cdot \left(\frac{1-p}{p}\right)^i$$

$$\left.\begin{array}{l} P_0 = 0 \\ P_N = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a + b = 0 \\ a + b \cdot (\frac{q}{p})^N = 1. \end{array}\right\} \Rightarrow 1 = b\left[(\frac{q}{p})^N - 1\right]$$

$q \triangleq 1 - p$

$$\Rightarrow b = \frac{1}{(\frac{q}{p})^N - 1}$$

$$a = -b.$$

$$\Rightarrow P_i = \frac{1}{1 - (\frac{q}{p})^N}\left(1 - (\frac{q}{p})^i\right)$$

$$\Rightarrow P_i = \begin{cases} \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^N} & \text{if } p \neq \frac{1}{2} \\ \frac{i}{N} & \text{if } p = \frac{1}{2} \end{cases}$$

If $p = \frac{1}{2}$

Induction $P_i = i P_1$

$$P_{i+1} = 2P_i - P_{i-1}. \quad P_{i+1} = 2 \cdot i P_1 - (i-1) P_1$$

$$= (i+1) P_1$$

$$P_N = N P_1 = 1 \Rightarrow P_1 = \frac{1}{N}.$$

# Drug Testing

New Drug. Cure rate $P_1$.

Decide whether $P_1 > P_2$ or $P_1 < P_2$

$(X_1, Y_1)$, ---, $(X_j, Y_j)$.

$X_i$ - drug 1

$Y_i \to$ drug 2.

$$Z_j = X_j - Y_j \qquad S_n = \sum_{j=0}^{n} Z_j$$

Test $S_n = M$ or $S_n = -M$ first.

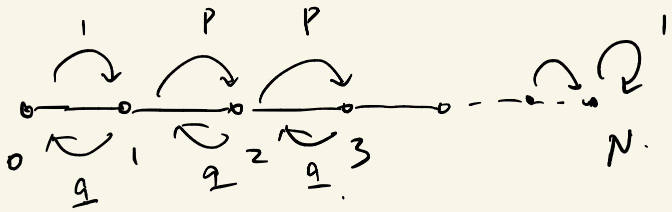
$$Z_j = \begin{cases} 1 & p_1(1-P_2) \\ -1 & P_2(1-P_1) \\ 0 & \text{o.w.} \end{cases} \qquad \boxed{\frac{q}{p} = \frac{P_2(1-P_1)}{P_1(1-P_2)}}$$

$$p = \frac{P_1(1-P_2)}{P_1(1-P_2)\,P_2(1-P_1)} \qquad q = 1-p.$$

$$P[P_2 > P_1] = 1 - \frac{1-(q/p)^M}{1-(q/p)^{2M}} = \frac{(q/p)^M - (q/p)^{2M}}{1-(q/p)^{2M}}$$

$$= \frac{(q/p)^M}{1+(q/p)^M} = \frac{1}{(p/q)^M + 1}.$$

Another random work on $\mathbb{N}$




$$h_i = E_i(N).\qquad \begin{cases} h_0 = 1 + h_1 \\ h_N = 0 \\ h_i = q\,h_{i-1} + p\,h_{i+1} + 1, & i \geqslant 1. \end{cases}$$

$Y_i \triangleq$ # of steps from $i$ to $i+1$.

$$N_{0,n} = \sum_{i=0}^{n-1} Y_i$$

$$y_i \triangleq E[Y_i] \Rightarrow \forall i > 1,\quad y_i = 1 + q\left(y_{i-1} + y_i\right).$$

$$\Rightarrow y_i = \frac{1}{p} + \frac{q}{p}\, y_{i-1}.$$

$$y_0 = 1.$$

$$y_0 = 1$$

$$y_1 = \frac{1}{p} + \frac{q}{p} = \frac{1}{p} + \alpha$$

$$y_2 = \frac{1}{p} + \alpha\left(\frac{1}{p} + \alpha\right) = \frac{1}{p} + \frac{1}{p}\alpha + \alpha^2$$

$$y_3 = \frac{1}{p} + \alpha\left(\frac{1}{p} + \frac{1}{p}\alpha + \alpha^2\right) = \frac{1}{p} + \frac{1}{p}\alpha + \frac{1}{p}\alpha^2 + \alpha^3$$

$$y_i = \frac{1}{p}\sum_{j=0}^{i-1} \alpha^j + \alpha^i$$

$$\alpha = 1 \Rightarrow y_i = 2i+1.$$

## 2SAT.

Random walk on solution space.

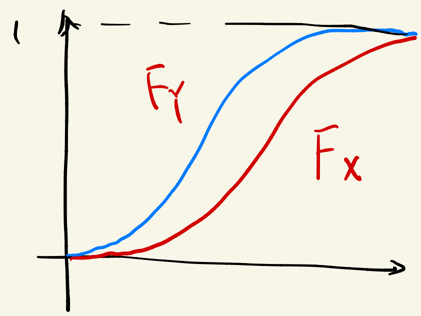
## Stochastic dominance.

$X \sim \mu,$

$Y \sim \nu.$

$\mu \succsim \nu$ if $\forall a \in \mathbb{R},$

$\mu[(a,+\infty)] \geq \nu[(a,+\infty)]$




Ex. RW on $\mathbb{N}$ $\succsim$ 2SAT algo.

Ex. $p \geq q \Rightarrow Ber(n,p) \succsim Ber(n,q).$

Ex. Connectedness on Erdős-Rényi.

<u>Coupling</u>. two distribution $\mu, \nu$ on $\Omega$.

$C$ joint of $\mu, \nu$ such that

$\forall (X, Y) \sim (\mu, \nu)$.

$X \sim \mu$.

$Y \sim \nu$.

<u>Example</u>. $Ber(n, p)$, $Ber(n, q)$.

<u>Thm</u>. $X \succcurlyeq Y$ iff $\exists$ Coupling $C$.

$$\underset{(X,Y)\sim C}{P}[X \geq Y] = 1.$$

pf. " if "

$$P[Y > a] = \underset{(X,Y)\sim C}{P}[Y > a]$$

$$= \underset{(X,Y)\sim C}{P}[X \geq Y > a]$$

$$\leq \underset{(X,Y)\sim C}{P}[X \geq a] = P[X > a].$$

" only if " Exercises.