Lec 3

$P_x(T_x < \infty) = 1$

$\Updownarrow$

Review.:

Irreducibility (I), aperiodicity (A), $\boxed{\text{recurrence (R)}}$ $\exists$ stationary (S).

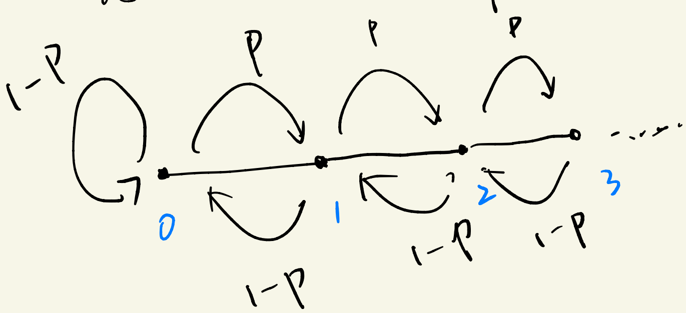
$\exists^{=1}$ stationary (U).

FTMC

For finite chains.

$$(I) + (A) \Rightarrow \text{unique stationary, convergence.}$$

How about infinite space?



$$\lambda(0) = (1-p)\lambda(0) + (1-p)\lambda(1) \Rightarrow \lambda(0) = \frac{1-p}{p}\lambda(1).$$

$$\lambda(1) = \lambda(0)p + (1-p)\lambda(2) \Rightarrow \lambda(1) = \frac{1-p}{p}\lambda(2)$$

$$\lambda(0) = \left(\frac{1-p}{p}\right)^{i}\lambda(i).$$

$$\begin{cases} \lambda(0) = \left(\frac{1-p}{p}\right)^{i}\lambda(i) \\ \sum_{i=0}^{\infty}\lambda(i) = 1 \end{cases}$$

Case 1 $p < \frac{1}{2}$:
$$\begin{cases} \pi(i) = \left(\frac{1-p}{p}\right) \pi(i+1) \\ \sum \pi(i) = 1. \end{cases}$$

$$\Rightarrow \quad \pi(i) = \left(\frac{p}{1-p}\right)^i \cdot c.$$

$$c \sum_{i=0}^{\infty} \left(\frac{p}{1-p}\right)^i = \frac{1-p}{1-2p} \cdot c = 1 \Rightarrow c = \frac{1-2p}{1-p}$$

There is a stationary!

Case 2 $p > \frac{1}{2}$. no stationary. Case 3 $p = \frac{1}{2}$.
No stationary.

For countable chain.
FIMC requires (I) + (A) + (S).

Review of Convergence.

$\{X_n\}$. X.

① Converge in Probability. $X_n \xrightarrow{P} \mu$.

$\forall \varepsilon$: $\Pr[\, |X_n - X| > \varepsilon \,] \to 0.$

② Converge with Probability 1 / Almost surely converge.

$X_n \xrightarrow{a.s} X$: if $\exists M \subseteq \Omega$ 1) $P[M] = 1$.
2) $\forall \omega \in M$, $X_n(\omega) \to X(\omega)$.

② $\Rightarrow$ ①.

$X_1, X_2, \dots,$ iid with mean μ. $\mathbb{E}[|X_i|] < \infty$

$$\bar{X}_n \triangleq \frac{1}{n} \sum_{i=1}^{n} X_i$$

WLLN: $\bar{X}_n \xrightarrow{P} \mu$.

SLLN: $\bar{X}_n \xrightarrow{a.s} \mu. \iff \Pr\left[\lim_{n \to \infty} \bar{X}_n = \mu \right] = 1.$

SLLN for MC.

Recall T_j is the first hitting time of j

$X_0, X_1, \dots$ a M.C starting at $X_0 = i$. $i \leftrightarrow j$.

then $$P_i\left[\lim_{n \to \infty} \frac{1}{n} \boxed{\sum_{t=1}^{n} \mathbb{1}[X_t = j]} = \frac{1}{\mathbb{E}_j T_j} \right] = 1.$$

~~Cor. (I) + (A) + (S)~~

$$\cancel{\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} P^t(i,j) = \frac{1}{\mathbb{E}_j(T_j)}}$$

~~Prop.~~ $\cancel{a_n \to a \Rightarrow \frac{1}{n} \sum_{t=1}^{n} a_t \to a.}$

[Cesaro Sum]

$$\cancel{\Rightarrow \pi(j) = \frac{1}{\mathbb{E}_j[T_j]}}.$$

Pf of SLLN

① j is transient $\Rightarrow$ $\mathbb{E}_j T_j = \infty$

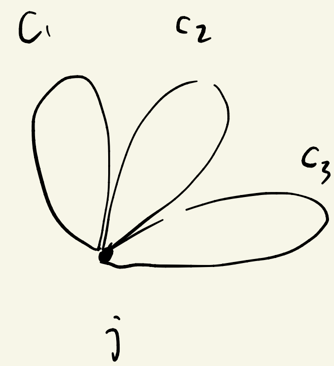
$$\Downarrow$$

$$\mathbb{E}_j[N_j] < \infty$$

$$\Downarrow$$

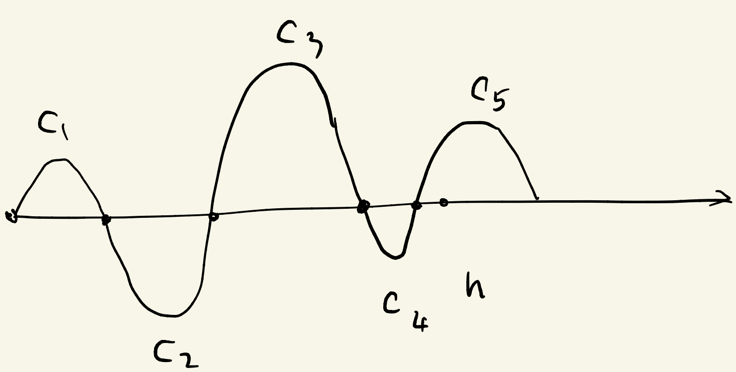
w.p 1 $N_j < \infty$.

② Assume $i = j$



$$S_k = C_1 + C_2 + \cdots + C_k$$

C_i : iid with $\mathbb{E}[C_i] = \mathbb{E}_j T_j$.



$$V_n \triangleq \max_k \{ S_k \le n \} \Rightarrow S_{V_n} \le n \le S_{V_n+1}$$

$$\Leftrightarrow \quad \frac{S_{V_n}}{V_n} \le \frac{n}{V_n} \le \frac{S_{V_n+1}}{V_n}$$

$n \to \infty$: $V_n \to n$

$$\frac{S_{V_n}}{V_n} \xrightarrow{\text{SLLN}} \mathbb{E}_j T_j \quad \Rightarrow \quad \frac{n}{V_n} \to \mathbb{E}_j T_j .$$

If $i \neq j$. $P_i\{T_j < \infty\} = 1 \Rightarrow$ the finite path $i \to j$ is negligible.

Thm. If $X_n \to c$ w.p. 1. $\forall n$: $X_n \in I$.

$\Rightarrow E[X_n] \to c$. (Bounded Convergence Thm).

(I) $\Rightarrow$

Cor. $\lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} p^t(i,j) = \frac{1}{E_j[T_j]}$. $\forall i,j$.

Cor $(I) + (A) + (S) \overset{FTMC}{\Rightarrow} \lim_{n\to\infty} p^n(i,j) \to \pi(j)$.

$$\overset{Cesaro}{\Rightarrow} \lim_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} p^t(i,j) \to \pi(j)$$

$$= \frac{1}{E_j[T_j]}.$$

Recall Recurrence $P_i[T_i < \infty] = 1$.

① positive recurrence. $\mathbb{E}_i[T_i] < \infty$

$\Rightarrow$ $\exists$ stationary. $\pi(i) = \dfrac{1}{\mathbb{E}_i[T_i]}$.

② null recurrence. $\mathbb{E}_i[T_i] = \infty$.

$$\boxed{\begin{array}{l} \text{Thm: Assuming (I).} \\ \text{positive recurrence} \Leftrightarrow (S) + (U) \end{array}}$$

pf: $\Rightarrow$.

(U). if π is stationary

$$\forall t, j: \sum_i \pi(i) P^t(i,j) = \pi(j)$$

$$\Rightarrow \sum_i \pi(i) \frac{1}{w} \sum_{t=1}^{w} P^t(i,j) = \pi(j)$$

$$\Rightarrow \pi(j) = \frac{1}{\mathbb{E}_j T_j}.$$

$$(S): \quad \frac{1}{n}\sum_{t=1}^{n} p^t(i,j) \to \frac{1}{E_j(T_j)}$$

If S is finite.

$$\sum_{j\in S} \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} p^t(i,j) = \sum_{j\in S} \frac{1}{E_j(T_j)}$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \sum_{j\in S} p^t(i,j) = 1.$$

$$P^t P = P^{t+1}$$

$$\Leftrightarrow \forall i, j: \quad \sum_k p^t(i,k) \cdot p(k,j) = p^{t+1}(i,j).$$

$$\Rightarrow \quad \sum_k \left[\frac{1}{n}\sum_{t=1}^{n} p^t(i,k)\right] \cdot p(k,j) = \frac{1}{n}\sum_{t=1}^{n} p^{t+1}(i,j)$$

$$\Rightarrow \quad \sum_k \pi(k)\, p(k,j) = \pi(j).$$

The case for infinit S is similar, but needs some more effort to deal with ∞.

" $\Leftarrow$ " is trivial.