Discrete Markov Chains

$X_0, X_1, \dots, X_t, \dots \quad X_i \in \Omega$

$\forall t \geq 1,$

$$\Pr[X_t = a_t \mid X_{t-1} = a_{t-1}, \dots, X_0 = a_0]$$
$$= \Pr[X_t = a_t \mid X_{t-1} = a_{t-1}]$$

Example Gambler's ruin.

1-D random walk.

* $X_i \sim \pi_i$. $\quad \pi_{i+1}^T = \pi_i \cdot P.$

$$p(i,j) = \Pr[X_{t+1} = j \mid X_t = i]$$

* stationary $\quad \pi^T P = \pi^T.$

Example random walk on G.

* MCMC. Stationary is the desired distribution.

<u>Card shuffling</u>. / volumn estimation.

Q1. $\exists$ stationary

Q2. $\exists^{=1}$?

Q3 Converge.

<u>Finite case</u>.

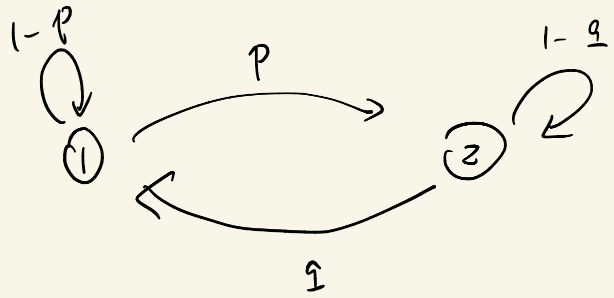
① <u>Perron - Frobenius Thm</u>

$\forall$ nonnegative matrix A has a nonnegative real eigenvalue a with spectral radius $\rho(A) = a$, and a has a nonnegative eigenvector.

$$P \cdot \mathbb{1} = \mathbb{1}. \Rightarrow P^T \text{ has eigenvalue } 1$$

$$\Rightarrow P^T x = x \quad \text{for } x \geq 0$$

②. ③. Case of two chains



$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}.$$

$$\lambda = \left[\frac{q}{p+q}, \frac{p}{p+q} \right]^T$$ is a stationary distr.

$$\Delta_t \triangleq \left| \mu_t(1) - \lambda(1) \right|$$

$$= \left| \mu_{t-1}^T \cdot P(1) - \lambda(1) \right| = \left| (1-p)\mu_{t-1}(1) + q\,(1-\mu_{t-1}(1)) - q/(p+q) \right|$$

$$= \left| (1-p-q)\,\mu_{t-1}(1) + q \cdot \left(1 - \frac{1}{p+q}\right) \right|$$

$$= \left| 1-p-q \right| \cdot \Delta_{t-1}.$$

$\Delta_t \to 0$ except.

1. $p = q = 0$
2. $p = q = 1$.

__Case 1.__ $p = q = 0$

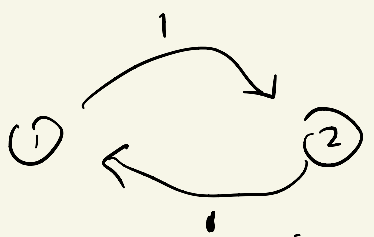
__reducible__

Stationary distribution might not be unique.

"j is accessible from i". equivalent relation.

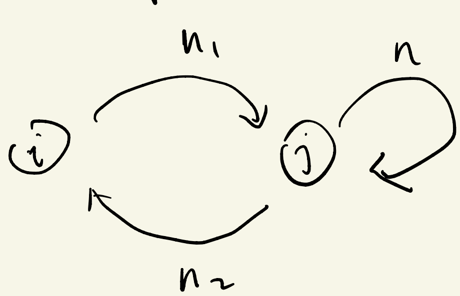
irreducible $\Leftrightarrow$ equivalent class $= 1$.

__Case 2.__ $p = q = 1$.



$\forall i \in \Omega$: $d_i = \gcd(n : P^n(i,i) > 0)$.

__Lem.__ if i, j communicate, then $d_i = d_j$.



$$\left.\begin{array}{l} d_i \mid n_1 + n_2 \\ d_i \mid n_1 + n_2 + n \end{array}\right\} \Rightarrow d_i \mid n.$$

aperiodic $\Leftrightarrow$ $\forall i$, $d_i = 1$.

Fundamental Theorem of Markov chains

<u>Thm</u>. If a finite MC P is irreducible and aperiodic, then there exists a unique stationary distr. π, and

$$\forall \mu: \lim_{t \to \infty} \mu^T P^t \to \pi^T.$$

Countable case

Recurrence (常返).

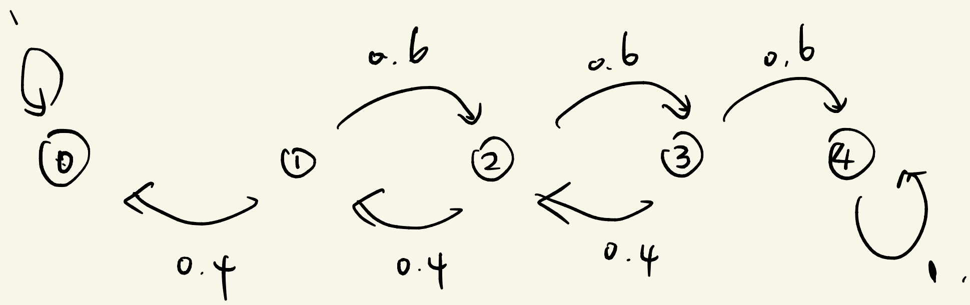
first hitting time $T_i = \inf \{ t > 0 : X_t = i \}$.

<u>Def</u>. i is <u>recurrent</u> if $P_i(T_i < \infty) = 1$.

i is <u>transient</u> if it is not recurrent.

$P_x(A) : \Pr[A \mid X_0 = x]$

Gambler's Ruin



0 4 : recurrent

1 2 3 : transient.

Thm. If i is recurrent, and j is accessible from i, then

① $P_i[T_j < \infty] = 1$

② $P_j[T_i < \infty] = 1$.

③ j is recurrent.

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pf.

① $i \xrightarrow{n} j$

$$I_n = \bigcup_{k=T_{n-1}+1}^{T_n} [X_k = j]. \qquad P_i[I_1] = p > 0$$

$$P_r[\exists n.\ I_n] = 1 - (1-p)^n \to 1.$$

②. if $P_j[T_i < \infty] = 1 - q < 1$.

w.p q. never come to i from j.

$\Rightarrow$ w.p. $p'q$, never come to i from i

$i \xrightarrow{p'} j$

③. $\Leftarrow$ ① + ②

Cor. finite irreducible chain every state is recurrent.

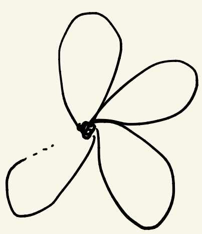
$N_i \triangleq \sum_{t=\infty}^{\infty} \mathbb{1}[X_t = i]$.

<u>Thm</u>. i is recurrent $\Leftrightarrow$ $\mathbb{E}_i[N_i] = \infty$.

Pf. " $\Rightarrow$ " Clearly $\mathbb{P}_i[N_i = \infty] = 1$.

so $\mathbb{E}_i[N_i] = \infty$.

" $\Leftarrow$ " if i is trasient. $\Pr[T_i = \infty] = \underline{q} > 0$.



each cycle, with prob. $\underline{q}$ never ends.

Geometric with $\underline{q}$. $\mathbb{E}[N_i] = 1/\underline{q}$.

Example. "A Drunk man will find his way home, but a Drunk bird may get lost forever".

$$S_n = \sum_{t=1}^{n} X_t, \quad X_t \in_R (\pm 1, \pm 1, \cdots, \pm 1) = \{\pm 1\}^d.$$

$d=1$: Calculate $\mathbb{E}_0[N_0] = \sum_n P^n(0,0)$

$n = 2m$. $\quad p^n(0,0) = \binom{2m}{m} \cdot 2^{-2m}$

Stirling: $k! \approx \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$

$$\binom{2m}{m} 2^{-2m} = \frac{(2m)!}{(m!)^2} \cdot 2^{-2m} \approx \frac{\sqrt{4\pi m} \cdot \left(\frac{2m}{e}\right)^{2m}}{2\pi m \cdot \left(\frac{m}{e}\right)^{2m}} \cdot 2^{-2m}$$

$$= \frac{1}{\sqrt{\pi m}} (2m)^{2m} \left(\frac{1}{m}\right)^{2m} \cdot 4^{-m}.$$

$$= \frac{1}{\sqrt{\pi m}}.$$

$$\Rightarrow \quad \sum_n p^n(0,0) = \sum_m p^{2m}(0,0) = \frac{1}{\sqrt{\pi}} \cdot \sum \frac{1}{\sqrt{m}} = \infty.$$