Lec 14

Def. A <u>standard Brownian Motion</u> $\{W(t) : t \geq 0\}$ is a stochastic process having

① continuous paths

② stationary, independent increment

③. $W(t) \sim N(0, t)$.

$W(t)$. Wiener Process.

Another characterization of SBM

A continuous random process $W(t)$ is a Brownian Motion iff.

Gaussian Process. $N(\mu, \Sigma)$.

① $\forall n > 0$, $t_1 < t_2 < \cdots < t_n$.

$(W(t_1), \ldots, W(t_n))$ is a joint Gaussian $\sim e^{\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}$

②. $E[W(t)] = 0$

③. $\forall s < t$, $Cov(X(s), X(t)) = s \wedge t$.

Pf. " $\Rightarrow$ "

$$a W(s) + b W(t)$$

$$= a W(s) + b \left(W(t) - W(s) + W(s) \right)$$

$$= (a+b) W(s) + b \left(W(t) - W(s) \right).$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]$$

$$= \mathbb{E}[XY]$$

$$\mathrm{Cov}(W(s), W(t))$$

$$= \mathbb{E}[W(s) W(t)]$$

$$= \mathbb{E}\left[W(s) \left(W(t) - W(s) + W(s) \right) \right]$$

$$= \mathbb{E}[W^2(s)] + \mathbb{E}[W(s) \cdot (W(t) - W(s))]$$

$$\underline{W(0)}, \quad W(t).$$

$$= s.$$

" $\Leftarrow$ " $\mathrm{Var}(W(s)) = s \Rightarrow W(s) \sim N(0, s).$

$$s = \mathrm{Cov}(W(s), W(t)) = \mathrm{Var}(W(s)) + \mathrm{Cov}(W(s), W(t) - W(s))$$

$$= s$$

$$\Rightarrow W(s) \perp W(t) - W(s).$$

Example W is a SBM $\Rightarrow$ $X(t) = tW(1/t)$ $\big(X(0)=0\big)$

is a SBM.

Pf. ①

$$\sum a_i X(t_i) = \sum a_i t_i W(1/t_i)$$ is Gaussian.

② $\mathbb{E}[X(s)] = s\,\mathbb{E}[W(1/s)] = 0.$

③ $\mathrm{Cov}(X(s), X(t))$ $\qquad$ $\mathrm{Cov}(aX, Y)$

$= \mathrm{Cov}(sW(1/s), t\,W(1/t)).$

$= st \cdot \frac{1}{t} = s.$

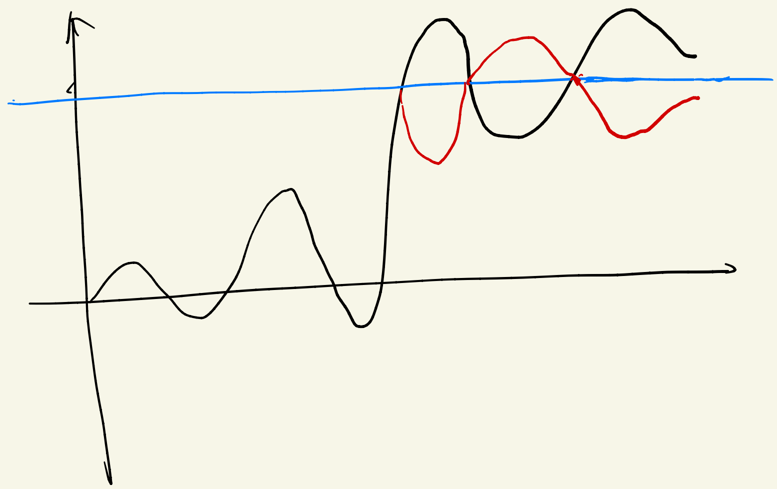
Def. A (μ, σ^2) Brownian motion is

$$X(t) = X(0) + \mu t + \sigma W(t).$$

where $W(t)$ is a SBM.

The reflection Principle

$W(t)$ SBM. $\tau_b = \inf\{t : W(t) > b\}$.



$$\Pr[\tau_b \leq t] = \Pr[\tau_b \leq t, W_t < b] + \Pr[\tau_b \leq t, W_t \geq b]$$

$$= \Pr[W_t < b \mid \tau_b \leq t] \Pr[\tau_b \leq t] + \Pr[W_t > b]$$

$$= \frac{1}{2} \Pr[\tau_b \leq t] + \Pr[W_t > b]$$

$$\Rightarrow \Pr[\tau_b \leq t] = 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)$$

The Brownian Bridge.

What is the distribution of $W(t)$ conditioned on $W(u)$?

∴ Gaussian ∵ by property of Gaussian vectors.

prop. $W(t) - (t/u)\, W(u) \perp W(u)$.

Pf. $\mathrm{Cov}\left(W(t) - t/u \cdot W(u),\ W(u) \right)$

$$= \mathrm{Cov}\left(W(t), W(u)\right) - t/u \,\mathrm{Var}\left(W(u)\right)$$

$$= t - t/u \cdot u = 0.$$

Therefore.

$$0 = \mathbb{E}\left[W(t) - t/u\, W(u) \right]$$

$$= \mathbb{E}\left[W(t) - \frac{t}{u} W(u) \,\middle|\, W(u) \right]$$

$$= \mathbb{E}\left[W(t) \,\middle|\, W(u) \right] - (t/u)\, W(u)$$

$$\Rightarrow \mathbb{E}\left[W(t) \,\middle|\, W(u) \right] = \frac{t}{u} W(u).$$

$$\mathrm{Var}\left[W(t) \,\middle|\, W(u) \right]$$

$$= \mathbb{E}\left[\left(W(t) - \mathbb{E}[W(t) | W(u)] \right)^2 \,\middle|\, W(u) \right]$$

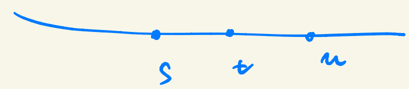
$$= \mathbb{E}\left[\left(W(t) - \frac{t}{u} W(u) \right)^2 \,\middle|\, W(u) \right]$$

$$= \mathbb{E}\left[\left(W(t) - \frac{t}{u} W(u) \right)^2 \right]$$

$$= t - 2(t/u)(t \wedge u) \cdot (t/u)^2 \cdot u$$

$$= \frac{t(u-t)}{u}$$

Standard Brownian Bridge.



SBM conditioned on $W(1)=0$.

prop. $\forall\ 0 \le s < t \le 1$. $\quad \mathrm{Cov}(W(s), W(t) \mid W(1)=0) = s(1-t)$

pf. $\mathbb{E}[W_s W_t \mid W_u] = \int_{\mathbb{R}} y \cdot \mathbb{E}[W_s \mid W_t = y, W_u] \cdot p_{W_t}(y \mid W_u)\, dy$

$$= \int_{\mathbb{R}} y^2 \cdot \frac{s}{t}\, p_{W_t}(y \mid W_u)\, dy = \frac{s}{t}\, \mathbb{E}[W_t^2 \mid W_u]$$

$$\mathrm{Var}[W(t) \mid W(u)] = \mathbb{E}[W(t)^2 \mid W(u)] - \left(\mathbb{E}[W(t) \mid W(u)]\right)^2$$

$$\Rightarrow \mathbb{E}[W^2(t) \mid W(u)] = \frac{t(u-t)}{u} + \left(\frac{t}{u} W(u)\right)^2$$

$$\mathrm{Cov}(W(s), W(t) \mid W(u)) = \mathbb{E}\Big[\big(W(s) - \mathbb{E}[W(s) \mid W(u)]\big)\big(W(t) - \mathbb{E}[W(t) \mid W(u)]\big) \Big| W(u)\Big]$$

$$= \mathbb{E}[W(s)W(t) \mid W(u)] - 2\,\mathbb{E}[W(s) \mid W(u)]\,\mathbb{E}[W(t) \mid W(u)] + \mathbb{E}[W(s) \mid W(u)]\,\mathbb{E}[W(t) \mid W(u)]$$

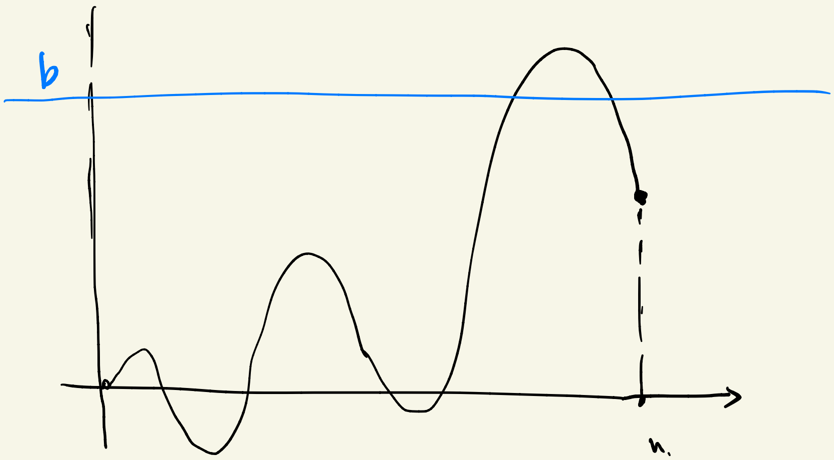
$$= \frac{s(u-t)}{u} + \frac{st}{u^2} W(u)^2 - \frac{s}{u} \cdot \frac{t}{u} W(u)^2 = \frac{s(u-t)}{u}.$$

Converse is also true.

$$\mathrm{Cov}(X\,Y \mid Z) = \mathbb{E}[XY \mid Z] - \mathbb{E}[X \mid Z]\,\mathbb{E}[Y \mid Z].$$

Construction of SBB. $\quad X(t) = W(t) - tW(1).$

Boundary Crossing Property



$$\Pr\left[\tau_b \le t \;\middle|\; W(t) = x\right]$$

$$= \frac{P[\tau_b \le t,\ W(t) \in dx]}{P[W(t) \in dx]} = \frac{P[\tau_b \le t]\, P[W(t) \in dx \mid \tau_b < t]}{\frac{1}{\sqrt{t}}\,\varphi\left(\frac{x}{\sqrt{t}}\right)}$$

$$= \frac{P[\tau_b \le t]\, P[W(t) \in 2b - dx \mid \tau_b < t]}{\frac{1}{\sqrt{t}}\,\varphi\left(x/\sqrt{t}\right)} = \frac{P[W(t) \in 2b - dx \wedge \tau_b \le t]}{\frac{1}{\sqrt{t}}\,\varphi\left(x/\sqrt{t}\right)}$$

$$\varphi(s) = e^{-\frac{s^2}{2}}$$

$$= \frac{P[W(t) \in 2b - dx]}{\frac{1}{\sqrt{t}}\,\varphi\left(x/\sqrt{t}\right)} = \frac{\frac{1}{\sqrt{t}}\,\varphi\left(\frac{2b-x}{\sqrt{t}}\right)}{\frac{1}{\sqrt{t}}\,\varphi\left(x/\sqrt{t}\right)}$$

$$= \exp\left(-\frac{1}{2}\frac{(2b-x)^2}{t} + \frac{1}{2}\frac{x^2}{t}\right)$$

$$= \exp\left(\frac{-2b(b-x)}{t}\right)$$

Testing for uniformity

$U_1, \dots U_n \sim F \in [0,1]$. Test $F(t) = t$.

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I[U_i \le t] \quad \text{for } 0 \le t \le 1.$$

Kolmogorov-Smirnov Test

reject if $F_n(t) - t$ is large for some t.

How to choose the threshold. ? S.L.L.N $F_n(t) - t \to 0$.

F_n is the sum of n ind. variables.

$$Var(I(U_1 \le t)) = t - t^2$$

C.L.T. $\sqrt{n}\,(F_n(t) - t) \xrightarrow{D} N(0, t(1-t))$.

$$X_n(t) = \sqrt{n}\,(F_n(t) - t).$$

$$Cov(1_{\{U_1 \le s\}}, 1_{\{U_1 \le t\}}) = s(1-t)$$

Vector C.L.T $\quad X(t)$ - SBB.

$$\begin{pmatrix} X_n(t_1) \\ \vdots \\ X_n(t_k) \end{pmatrix} \xrightarrow{D} \begin{pmatrix} X(t_1) \\ \vdots \\ X(t_k) \end{pmatrix}.$$

$*.\quad \lim_{n\to\infty} P\{\max X_n(t) : t \in [0,1] \geq b\}$

$= P[\max X(t) \geq b] = P[\tau_b \leq 1] = e^{-2b^2}$