Lec 13

Convergence of Martingale.

Prop. $X_0, X_1, \dots$ nonnegative supermartingale. $X_0 \le a$.

$\forall b > a$, $T_b \triangleq \inf\{t : X_t \geq b\}$. $\Pr[T_b < \infty] \le a/b$.

Pf. $\forall t > 0$.

$$\mathbb{E}[X_{T \wedge t}] \le \mathbb{E}[X_0] \le a.$$

if $T \le t$. $X_{T \wedge t} \geq b$.

$$\Rightarrow \quad X_{T \wedge t} \geq b \mathbb{1}[T \le t].$$

$$\Rightarrow \quad a \geq \mathbb{E}[X_{T \wedge t}] \geq b \cdot \Pr[T \le t].$$

$$\Rightarrow \quad \Pr[T \le t] \le a/b.$$

$$\overset{\exists t:}{\Rightarrow} \quad P[T < \infty] < P[T \le t] \le a/b.$$

OST for supermartingale.

if OST conditions are satisfied. then $\mathbb{E}[X_S] \le \mathbb{E}[X_T]$.

Thm. A nonnegative supermartingale converges w.p. 1.

Pf. ① $\mathbb{E}[X_t] \leq \mathbb{E}[X_0]$. $\Rightarrow$ $X_t \overset{t\to\infty}{\not\to} \infty$.

② $0 \leq a < b$.

$$T_0 = 0.$$

$$S_1 = \inf\{t \geq T_0 : X_t \leq a\}.$$

$$T_1 = \inf\{t \geq S_1 : X_t \geq b\}.$$

$$\vdots$$

$$S_k = \inf\{t \geq T_{k-1} : X_t \leq a\}$$

$$T_k = \inf\{t \geq S_k : X_t \geq b\}.$$

$\forall\, n \in \mathbb{N}$. OST $\Rightarrow$ $\mathbb{E}[X_{T_k \wedge n} - X_{T_{S_k} \wedge n}] \leq 0$.

* If $T_k \leq n$, then $X_{T_k \wedge n} \geq b$.

* If $T_k > n$, then $X_{T_k \wedge n} = X_n$.

$$\Rightarrow \quad X_{T_k \wedge n} \geq b \cdot \mathbb{1}[T_k \leq n] + X_n \mathbb{1}[T_k > n]$$

$$X_{S_k \wedge n} \leq a \cdot \mathbb{1}[S_k \leq n] + X_n \mathbb{1}[S_k > n].$$

$$X_{T_k \wedge n} - X_{S_k \wedge n} \geq b \cdot \mathbb{1}[T_k \leq n] - a\, \mathbb{1}[S_k \leq n] + X_n\left(\mathbb{1}[T_k > n] - \mathbb{1}[S_k > n]\right)$$

$$\geq b\, \mathbb{1}[T_k \leq n] - a\, \mathbb{1}[S_k \leq n]$$

$$\Rightarrow \quad b \Pr[T_k \le n] - a \Pr[S_k \le n] \le 0.$$

$$\Rightarrow \quad \Pr[T_k < \infty] \le \left(\frac{a}{b}\right) \Pr[S_k < \infty] \le \Pr[T_{k-1} < \infty].$$

Stochastic Approximation

Goal. Find the zero of a function. but the value may contain noise.

Thm. $f : \mathbb{R} \to \mathbb{R}$. $\mathbb{E}[(X_0)^2] < \infty$.

$$Y_n = f(X_n) + y_n.$$

$$X_{n+1} = X_n - a_n Y_n.$$

If.

① $X_0, y_0, y_1, \ldots$ are independent, $\mathbb{E}[y_i] = 0$. $\mathrm{Var}[y_i] = 1$.

②. $|f(x)| \le c|x|$. for some $1 < c < \infty$.

③. $\forall \delta > 0 \quad \inf_{|x| > \delta} (x f(x)) > 0.$

④. $a_n \ge 0, \quad \sum a_n = \infty.$

⑤. $\sum a_n^2 < \infty.$

$X_n \to 0$ w.p. 1 as $n \to \infty$.

Pf. $\mathbb{E}[X_{n+1}^2 \mid \overline{X}_{0,n}]$

$$= \mathbb{E}\left[\left(X_n - a_n (f(X_n) + y_n) \right)^2 \mid \overline{X}_{0,n} \right]$$

$$= X_n^2 - 2a_n X_n f(X_n) + a_n^2 (f(X_n)^2 + 1).$$

$|f(x)| \leq c|x|$

$$\leq X_n^2 - 2a_n X_n f(X_n) + a_n^2 c^2 (X_n^2 + 1)$$

$$= X_n^2 (1 + a_n^2 c^2) - 2a_n X_n f(X_n) + a_n^2 c^2.$$

Define $W_n = b_n (X_n^2 + 1)$, $\quad b_n = \left(\prod_{k=1}^{n-1} (1 + a_k^2 c^2) \right)^{-1}$

$$\mathbb{E}[W_{n+1} \mid \overline{X}_{0,n}] = b_{n+1} \mathbb{E}[X_{n+1}^2 \mid \overline{X}_{0,n}] + b_{n+1}$$

$$\leq b_{n+1} \left(X_n^2 (1 + a_n^2 c^2) + a_n^2 c^2 \right) + b_{n+1}$$

$$= b_{n+1} (1 + a_n^2 c^2) (X_n^2 + 1) = b_n (X_n^2 + 1) = W_n.$$

$\{W_n\}$ is a nonnegative supermartingale.

$\prod (1 + a_n^2 c^2) \leq e^{c^2 \sum a_n^2}$ $\downarrow$. $\Rightarrow X_n^2 \downarrow$

What we have is

$$\mathbb{E}[W_{n+1} \mid \bar{X}_{0,n}] \le W_n - \underline{2a_n b_{n+1} X_n f(X_n)}. \quad > \varepsilon.$$

$\forall \delta > 0.$ $\quad D \triangleq \{ x : |x| > \delta \}.$

Bad event $B_m \triangleq \bigcap_{n \ge m} X_n \in D.$

$$\varepsilon = \inf_{x \in D} x f(x).$$

$$X_n f(X_n) \ge \mathbb{1}[X_n \in D] \cdot \varepsilon \qquad \text{for all } m \le n$$

$$\Rightarrow \quad \mathbb{E}[X_n f(X_n)] \ge \varepsilon \cdot P_r[X_n \in D] \ge \varepsilon P(B_m)$$

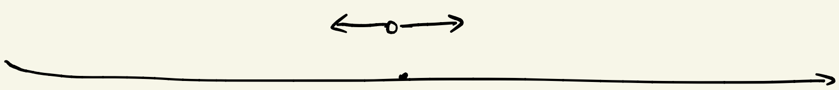
$$\Rightarrow \quad \mathbb{E}[W_{n+1}] \le \mathbb{E}[W_n] - 2a_n b_{n+1} \varepsilon P(B_m)$$

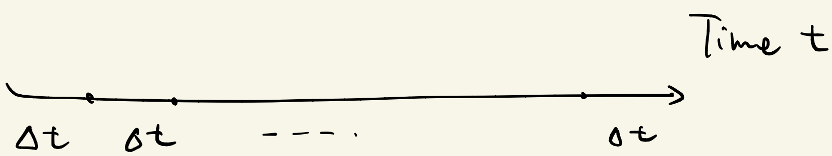
$$\le \mathbb{E}[W_m] - 2\varepsilon P(B_m) \sum_{k=m}^{n-1} a_k b_{k+1}.$$

$$\Rightarrow \quad P(B_m) \le \frac{\mathbb{E}[W_m]}{2\varepsilon \sum_{k=m}^{n-1} a_k b_{k+1}} \longrightarrow 0.$$

Brownian Motion

1-D Random work.





$$X(t) = \delta \cdot \left(X_1 + X_2 + \cdots + X_{t/\Delta t} \right).$$

$X_i \in \{+1, -1\}$.

$Var(X_i) = 1$.

$$\mathbb{E}[X(t)] = 0$$

$$Var(X(t)) = \frac{\delta^2}{\Delta t} \cdot t. \qquad \text{let } \delta = \sigma\sqrt{\Delta t}.$$

$$= \sigma^2 t.$$

<u>Thm</u>. $X_1, X_2, \dots$ a sequence of i.i.d. r.v. mean μ, variance σ^2.

$$\frac{\sum X_i - n\mu}{\sigma\sqrt{n}} \overset{n\to\infty}{\sim} N(0,1)$$

$$Pr\left[\frac{X_1 + \cdots X_n - n\mu}{\sigma\sqrt{n}} \leq a \right] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} \, dx.$$

Cor. When $\Delta t \to 0$. $Var(X_i) = \sigma^2 \Delta t$.

$E[X_i] = 0$.

$$X(t) \sim \sigma\sqrt{\Delta t}\sqrt{\frac{t}{\Delta t}} \cdot N(0,1) = N(0, \sigma^2 t).$$

$\forall\ t_1 < t_2 < \cdots < t_n$.

$X(t_n) - X(t_{n-1}),\ X(t_{n-1}) - X(t_{n-2}),\ \cdots,\ X(t_2) - X(t_1),\ X(t_1)$

are independent. $X(t+s) - X(t)$ only depends on s.

Def. $\{X(t), t \geq 0\}$ is said to be a <u>Brownian motion</u> process of.

① $X(0) = 0$.

②

③ $X(t) \sim N(0, \sigma^2 t)$.