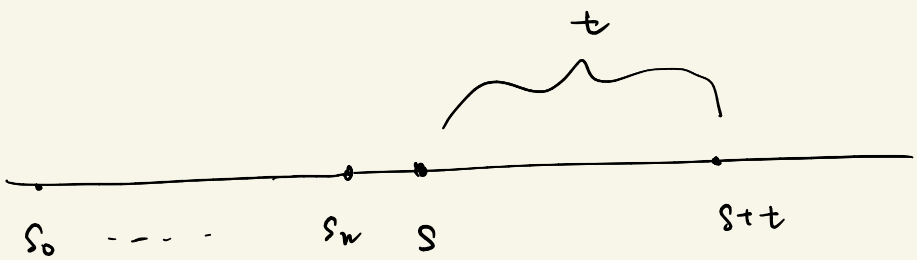
Lec 10

Continuous Time Markov chain.



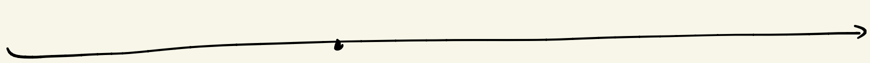
$$P[\,X_{s+t} = j \mid X_s = i,\ X_{s_n} = i_n,\ \dots,\ X_{s_0} = i_0\,]$$

$$= P_r[\,X_{s+t} = j \mid X_s = i\,] = P_r[\,X_t = j \mid X_0 = i\,].$$

Example ☆ $N(t)$, $t \geq 0$ a Poisson Process with rate λ.

Y_n. discrete time chain with transition $u(i,j)$.

$$X_t = Y_{N(t)}.$$



each arrival. transition according to $u(\cdot,\cdot)$.

Transition Probability

$\forall t.\quad P_t(i,j) \triangleq P_r[X_t = j \mid X_0 = i]$.

In ★. $\quad P_t(i,j) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} w^n(i,j)$.

Thm. (Chapman-Kolmogorov Equation).

$$P_{s+t}(i,j) = \sum_k P_s(i,k) P_t(k,j).$$

Pf. $P_r[X_{s+t} = j \mid X_0 = i] = \sum_k P_r[X_{s+t} = j, X_s = k \mid X_0 = i]$

$$= \sum_k P[X_{s+t} = j \mid X_s = k, X_0 = i] \cdot P_r[X_s = k \mid X_0 = i]$$

$$= \sum_k P_t(k,j) \cdot P_s(i,k).$$

P_{s+t} can be determined by smaller s and t.

so only infinitesimal time enough.

The jump rate.

$$q(i,j) \triangleq \lim_{h\to 0} \frac{P_h(i,j)}{h} \qquad \text{for } j \neq i$$

In example ☆.

Consider a time slot of length h. $\qquad e^{-\lambda h} \frac{(\lambda h)^n}{n!}$

0 jumps: $\quad e^{-\lambda h}$

1 jumps: $\quad e^{-\lambda h} \cdot \lambda h.$

≥ 2 jumps: $\quad 1 - e^{-\lambda h} - \lambda h e^{-\lambda h}$

$$= 1 - (1+\lambda h) e^{-\lambda h}$$

$$= 1 - (1+\lambda h)\left(1 - \lambda h + \frac{(\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \cdots\right).$$

$$= o(h).$$

$$\frac{P_h(i,j)}{h} \approx e^{-\lambda h} \lambda \, u(i,j) \to \lambda u(i,j).$$

Example. [Poisson Process].

$X(t)$. # of arrivials. up to time t. in Poisson(λ).

$\forall n$. $q(n, n+1) = \lambda$.

Poisson Race.

$S \sim \exp(\lambda)$

$T \sim \exp(\mu)$.

$\min\{S, T\} \sim \exp(\lambda+\mu)$.

Example (M/M/s Queue).

λ μ

$q(n, n+1) = \lambda$.

$$q(n, n-1) = \begin{cases} n\mu, & 0 \le n \le s \\ s\mu, & n \ge s. \end{cases}$$

Example Branching Process

Each individual dies at rate μ, gives birth at rate λ.

$q(n, n+1) = \lambda n$. $q(n, n-1) = \mu n$.

Q. How to construct the chain given rate.

A. As in example ☆

$$\lambda_i = \sum_{j \neq i} q(i,j). \qquad q(i,j) = \lambda_i \cdot \frac{q(i,j)}{\lambda_i}$$

Computing the Transition Probability.

By Chapman - Kolmogorov.

$$P_{t+h}(i,j) - P_t(i,j) = \left(\sum_k P_h(i,k) P_t(k,j) \right) - P_t(i,j)$$

$$= \underbrace{\left(\sum_{k \neq i} P_h(i,k) P_t(k,j) \right)}_{A} + \underbrace{\left(P_h(i,i) - 1 \right)}_{B} P_t(i,j).$$

check :

$$\lim_{h \to 0} \frac{1}{h} A = \underline{\sum_{k \neq i} \lim_{h \to 0}} \frac{1}{h} P_h(i,k) P_t(k,j)$$

$$= \sum_{k \neq i} q(i,k) P_t(k,j)$$

$$\lim_{h \to 0} \frac{1}{h} B = - \lim_{h \to 0} \sum_{k \neq i} \frac{P_h(i,k)}{h} = - \sum_{k \neq i} q(i,k) = - \lambda_i$$

$$\Rightarrow \quad P_t'(i,j) = \sum_{k \neq i} q(i,k) P_t(k,j) - \lambda_i P_t(i,j).$$

$$Q(i,j) \triangleq \begin{cases} q(i,j) & \text{if } j \neq i \\ -\lambda_i & \text{if } j = i \end{cases}$$

$$\Rightarrow \quad P_t' = Q P_t.$$ [Kolmogorov backward equation].

$$P_t = e^{Qt} = \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!} = \sum_{n=0}^{\infty} Q^n \cdot \frac{t^n}{n!}.$$

Verification.

$$\frac{d}{dt} e^{Qt} = \sum_{n=1}^{\infty} Q^n \frac{t^{n-1}}{(n-1)!} = Q \cdot e^{Qt}.$$

Similarly $[0, t+h] \to [0,t] + [t, t+h]$.

$$P_{t+h}(i,j) - P_t(i,j) = \left(\sum_k P_t(i,k) P_h(k,j) \right) - P_t(i,j)$$

$$\Rightarrow \quad P_t' = P_t Q.$$ [Kolmogorov Forward Equation]

Example [Poisson Process] $X(t)$ Poisson(λ).

$$j \geq i: \quad P_t(i,j) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}$$

$$P_t'(i,j) = \lambda P_t(i+1,j) - \lambda P_t(i,j)$$

$$\frac{d}{dt}\left[e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} \right] = -\lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{j-i}}{(j-i)!} + \frac{\lambda^{j-i}}{(j-i)!} (j-i) \cdot t^{j-i-1} \cdot e^{-\lambda t}$$

Example [Two States Chain].

$$q(1,2) = \lambda. \qquad Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}.$$

$$q(2,1) = \mu.$$

$$\begin{bmatrix} P_t'(1,1) & P_t'(1,2) \\ P_t'(2,1) & P_t'(2,2) \end{bmatrix} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \begin{bmatrix} P_t(1,1) & P_t(1,2) \\ P_t(2,1) & P_t(2,2) \end{bmatrix}.$$

$$\Rightarrow \quad P_t'(1,1) = -\lambda P_t(1,1) + \lambda P_t(2,1)$$

$$P_t'(2,1) = \mu P_t(1,1) - \mu P_t(2,1).$$

$$\Rightarrow \left(P_t(1,1) - P_t(2,1) \right)' = -(\lambda+\mu)\left[P_t(1,1) - P_t(2,1) \right]$$

$$\Rightarrow \quad P_t(1,1) - P_t(2,1) = e^{-(\lambda+\mu)t}$$

$$\Rightarrow \quad P_t'(1,1) = -\lambda e^{-(\lambda+\mu)t}$$

$$\Rightarrow \quad P_t(1,1) = P_0(1,1) - \lambda \int_0^t e^{-(\lambda+\mu)s} \, ds$$

$$= 1 + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)s} \Big|_0^t$$

$$= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\mu+\lambda)t}.$$

$$P_t(2,1) = \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda} e^{-(\mu+\lambda)t}.$$

Limiting Behavior.

X_t is irreducible if $i \to j$ in finite steps

π stationary if $\pi^T P_t = \pi^T$ for all $t>0$.

Lem. π is a stationary distribution $\iff \pi^T Q = 0$.

$$\lambda_j \pi(j) = \sum_{k \neq j} q(k,j) \cdot \pi(k).$$

Pf. if $\pi^T P_t = \pi^T$, then

$$0 = \frac{d}{dt} \pi P_t(j) = \sum_i \pi(i) p'(i,j) = \sum_i \pi(i) \sum_k P_t(i,k) Q(k,j)$$

$$= \sum_k \sum_i \pi(i) P_t(i,k) Q(k,j) = \sum_k \pi(k) Q(k,j).$$

If $\pi Q = 0$ Fix j

$$\frac{d}{dt}\left(\sum_i \pi(i) P_t(i,j) \right) = \sum_i \pi(i) P_t'(i,j)$$

$$= \sum_i \pi(i) \sum_k Q(i,k) P_t(k,j) = \sum_k \sum_i \pi(i) Q(i,k) P_t(k,j)$$

$$= 0.$$

Thm. If X_t is irreducible, and has a stationary distribution π

then $\lim_{t \to \infty} P_t(i,j) = \pi(j)$.

Detailed balance Condition.

$\pi(i)\, \underline{q}(i,j) = \pi(j)\, \underline{q}(j,i) \qquad j \neq i$

π is stationary dist.

pf. $\quad \pi^T Q(j) = \sum_{i\neq j} \pi(i)\, \underline{q}(i,j) - \lambda_j \pi(j)$

$= \sum_{i\neq j} \pi(j)\, \underline{q}(j,i) - \lambda_j \pi(j)$

$= \pi(j) \left(\sum_{i\neq j} q(j,i) - \lambda_j \right) = 0.$