AI2613 随机过程 2020–2021 春季学期

Lecture 14 – Brownian Motion (II)

2021 年 5 月 31 日

Lecturer: 张驰豪 *Scribe*: 杨宽

1 Another Characterization of Brownian Motion

Recall that we defined a (standard) Brownian motion in the last lecture. A Brownian motion $\{W(t)\}_{t\geq 0}$ is *standard* if $W(1)$ has a *standard* normal distribution.

Definition 1 (Standard Brownian Motion (Wiener Process)). A stochastic process $\{W(t)\}_{t\geq 0}$ is said to be a *standard Brownian motion* if

- $W(0) = 0$.
- **Independent increments.** $\forall\, 0 = t_0 < t_1 < \cdots < t_n$,

$$W(t_n) - W(t_{n-1}),\quad W(t_{n-1}) - W(t_{n-2}),\quad \ldots,\quad W(t_1) - W(t_0)$$

  are independent.
- **Stationary increments.** $\forall\, t, s > 0$, $W(t+s) - W(t)$ only depends on s, and has a normal distribution $\mathcal{N}(0, s)$ for some constant σ.
- $W(t)$ is continuous.

We now introduce another characterization of a standard Brownian motion. Recall the definition of high dimensional Gaussian distribution. Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a n-dimensional vector. Then X is said to be *Gaussian* if $\forall\, a_1, a_2, \ldots, a_n$,

$$a_1 \cdot X_1 + a_2 \cdot X_2 + \cdots + a_n \cdot X_n$$

has a (one-dimensional) Gaussian distribution. Another way to define high dimensional Gaussian is to give the probability density function. If there exists a n-vector μ and a symmetric, positive semidefinite $n \times n$ matrix Σ such that the probability density function can be written as

$$f(\bar{x}) = (2\pi)^{-n/2} \cdot |\det \Sigma|^{-1/2} \cdot \exp\left(-\frac{1}{2} \cdot (\bar{x} - \mu)^{\mathsf{T}} \Sigma^{-1} (\bar{x} - \mu)\right).$$

In fact, if X is a Gaussian, we obtain that

$$\Sigma = \left(\mathrm{Cov}\left(X_i, X_j\right)\right)_{i,j}, \quad \text{and} \quad \mu(i) = \mathbb{E}[X_i] .$$

To introduce another characterization of a standard Brownian motion, we first define the *Gaussian process.*

Definition 2 (Gaussian Process). A stochastic process $\{W(t)\}_{t\geq 0}$ is called a Gaussian process if $\forall\, 0 < t_1 < \cdots < t_n$,

$$\left(W(t_1), W(t_2), \ldots, W(t_n)\right)$$

is a Gaussian.

Then a standard Brownian motion could be described by a Gaussian process.

Definition 3. A stochastic process $\{W(t)\}_{t\geq 0}$ is called a standard Brownian motion if

1. $\{W(t)\}$ is a continuous Gaussian process;
2. $\forall\, s$, $\mathbb{E}[X(s)] = 0$;
3. $\forall\, s \leq t$, $\mathrm{Cov}(X(s), X(t)) = s$.

Remark. It is more clear to use Definition 3 to justify a stochastic process is a Brownian motion. The only difficulty is to compute the covariance.
We recall the definition of covariance here:

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] .$$

We now verify that Definition 1 and Definition 3 are equivalent.

Proof. We first assume that $\{W(t)\}$ is a standard Brownian motion defined by Definition 1. We now show that it satisfies all conditions in Definition 3.
Fix $s \leq t$. We first show that $(W(s), W(t))$ is a Gaussian vector. Note that $\forall\, a, b \in \mathbb{R}$,

$$\begin{aligned} a \cdot W(s) + b \cdot W(t) &= a \cdot W(s) + b \cdot (W(t) - W(s) + W(s)) \\ &= (a+b) \cdot W(s) + b \cdot (W(t) - W(s)) \end{aligned}$$

is the sum of two independent Gaussian variables. So $a \cdot W(s) + b \cdot W(t)$ has a Gaussian distribution as well. Next we compute $\mathrm{Cov}(W(s), W(t))$ as follows:

$$\begin{aligned} \mathrm{Cov}(W(s), W(t)) &= \mathrm{Cov}(W(s), W(t) - W(s) + W(s)) \\ &= \mathrm{Var}[W(s)] + \mathrm{Cov}(W(s), W(t) - W(s)) \\ &= s . \end{aligned}$$

Thus we justify that $\{W(t)\}$ satisfies all conditions in Definition 3.

We now show that a stochastic process $\{X(t)\}$ satisfying conditions in Definition 3 is a Brownian motion defined by Definition 1. Since $\forall\, 0 < t_1 < \cdots < t_n$,

$$\big(X(t_1), X(t_2), \ldots, X(t_n)\big)$$

is a Gaussian vector, it is clear that $X(t)$ has independent increments and $X(t) - X(s)$ has a Gaussian distribution. We also know that $\mathbb{E}[X(t)] = 0$, and $\mathrm{Var}[X(t)] = \mathrm{Cov}(X(t), X(t)) = t$, so it justifies that $\{X(t)\}$ is indeed a Brownian motion defined by Definition 1. $\square$

Example 4. Suppose $\{W(t)\}$ is a standard Brownian motion. We claim that

$$X(t) = t \cdot W(1/t), \qquad X(0) = 0$$

is also a standard Brownian motion.

We use Definition 3 to verify that $\{X(t)\}$ is a standard Brownian motion. First, for all $a_1, a_2, \ldots, a_n$ and $t_1 \le t_2 \le \cdots \le t_n$,

$$\sum a_i \cdot t_i \cdot W(1/t_i)$$

has a Gaussian distribution since $\{W(t)\}$ is a Gaussian process. Therefore $\{X(t)\}$ is also a Gaussian process. It is clear that for all $t \ge 0$, $\mathbb{E}[X(t)] = t \cdot \mathbb{E}[W(1/t)] = 0$. Hence it suffices to verify the covariance. Fix $s \le t$,

$$\begin{aligned}
\mathrm{Cov}(X(s), X(t)) &= \mathrm{Cov}(s \cdot W(1/s), t \cdot W(1/t)) \\
&= s \cdot t \cdot \mathrm{Cov}(W(1/s), W(1/t)) \\
&= s \cdot t \cdot 1/t \ = s\,.
\end{aligned}$$

Remark. In fact, a nonstandard Brownian motion could be generated by a standard Brownian motion.

Definition 5. $X(t)$ is a (μ, σ^2) Brownian motion if

$$X(t) = X(0) + \mu \cdot t + \sigma \cdot W(t)\,.$$

Example 6 (Hitting Time). We now consider the hitting time in a Brownian motion. Let

$$\tau_b \triangleq \inf\{t \ge 0 : W(t) > b\}\,.$$

Since $W(t)$ is continuous, it is clear that $W(\tau_b) = b$, and $\forall\, t < \tau_b$, $W(t) < b$.

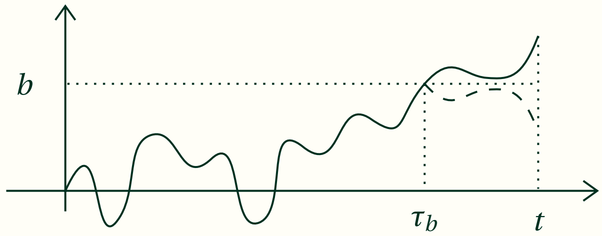

Figure 1: A hitting time and the reflection principle

Now, we are interested in the distribution of τ_b. Namely, for all $t > 0$, our goal is to compute $\mathbf{Pr}[\tau_b < t]$.

$$\begin{aligned}\mathbf{Pr}[\tau_b < t] &= \mathbf{Pr}[\tau_b < t \wedge W(t) > b] + \mathbf{Pr}[\tau_b < t \wedge W(t) < b] \\ &= \mathbf{Pr}[W(t) > b] + \mathbf{Pr}[W(t) < b \mid \tau_b < t] \cdot \mathbf{Pr}[\tau_b < t] \\ &= \mathbf{Pr}[W(t) > b] + \frac{1}{2} \cdot \mathbf{Pr}[\tau_b < t] \,.\end{aligned}$$

Here we use the *reflection principle*, which yields that for every $t' < t$, $\mathbf{Pr}\left[W(t) < b \mid \tau_b = t'\right] = \mathbf{Pr}\left[W(t) > b \mid \tau_b = t'\right]$ by symmetry. Therefore, it follows that

$$\mathbf{Pr}[\tau_b < t] = 2 \cdot \mathbf{Pr}[W(t) > b] = 2 \cdot \left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard Gaussian distribution, i.e.,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} \mathrm{e}^{-t^2/2} \, \mathrm{d}t \,.$$

2 Brownian Bridge

Consider a Brownian motion starting from $W(0) = 0$ and ending at $W(u) = x$.
Conditioned on fixed $W(0)$ and $W(u)$, an interesting question is to ask the distribution of $W(t)$. Intuitively, since $\{W(t)\}$ is a Brownian motion, we have $(W(t), W(u))$ is a Gaussian vector, and thus $W(t)$ itself is a Gaussian variable. So it is sufficient to compute its mean and variance. A reasonable conjecture is $t \cdot x / u$. We now rigorously prove our intuition.

Proposition 1. *$W(t) - (t/u) \cdot W(u)$ is independent of $W(u)$.*

Proof. We prove it by verifying that the covariance of the two random variables is 0. Note that

$$\begin{aligned} \mathrm{Cov}(W(t) - (t/u)\cdot W(u), W(u)) &= \mathrm{Cov}(W(t), W(u)) - (t/u)\cdot \mathrm{Cov}(W(u), W(u)) \\ &= t - (t/u)\cdot u \ = 0, \end{aligned}$$

which justifies the desired proposition. □

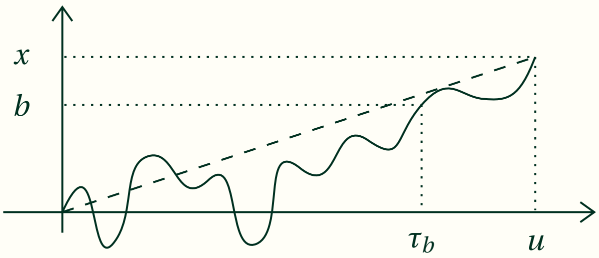

Figure 2: A Brownian bridge

We now compute the mean and variance of $W(t)$ by applying Proposition 1.
Since $\mathbb{E}[W(t)] = \mathbb{E}[W(u)] = 0$ in a Brownian motion, it follows that

$$\begin{aligned} 0 &= \mathbb{E}\left[W(t) - \frac{t}{u}\cdot W(u)\right] \\ &= \mathbb{E}\left[W(t) - \frac{t}{u}\cdot W(u) \mid W(u)\right] \\ &= \mathbb{E}[W(t) \mid W(u)] - \frac{t}{u}\cdot \mathbb{E}[W(u) \mid W(u)] \\ &= \mathbb{E}[W(t) \mid W(u)] - \frac{t}{u}\cdot W(u), \end{aligned}$$

which implies that

$$\mathbb{E}[W(t) \mid W(u)] = \frac{t}{u}\cdot W(u).$$

Next, we compute the variance of $W(t)$ conditioned on $W(u)$ as follows:

$$\begin{aligned} \mathrm{Var}[W(t) \mid W(u)] &= \mathbb{E}\left[\left(W(t) - \mathbb{E}[W(t) \mid W(u)]\right)^2 \mid W(u)\right] \\ &= \mathbb{E}\left[\left(W(t) - \frac{t}{u}\cdot W(u)\right)^2 \mid W(u)\right] \\ &= \mathbb{E}\left[\left(W(t) - \frac{t}{u}\cdot W(u)\right)^2\right] \\ &= \mathbb{E}\left[W(t)^2\right] - \frac{2t}{u}\cdot \mathbb{E}[W(t)\cdot W(u)] + \frac{t^2}{u^2}\cdot \mathbb{E}\left[W(u)^2\right] \\ &= t - \frac{2t^2}{u} + \frac{t^2}{u^2} \ = \frac{t(u-t)}{u}. \end{aligned}$$

Finally, to characterize the distribution of $\{W(t)\}$ conditioned on $W(u) = x$, we also need the covariances:

$$\begin{aligned}
\mathrm{Cov}(W(s), W(t) \mid W(u)) &= \mathbb{E}[W(s) \cdot W(t) \mid W(u)] - \mathbb{E}[W(s) \mid W(u)] \cdot \mathbb{E}[W(t) \mid W(u)] \\
&= \int_{\mathbb{R}} y \cdot \mathbb{E}\big[W(s) \mid W(t) = y, W(u)\big] \cdot p_{W(t)}\big(y \mid W(u)\big)\,\mathrm{d}y - \frac{s \cdot t}{u^2} \cdot W(u)^2 \\
&= \int_{\mathbb{R}} y \cdot \mathbb{E}\big[W(s) \mid W(t) = y\big] \cdot p_{W(t)}\big(y \mid W(u)\big)\,\mathrm{d}y - \frac{s \cdot t}{u^2} \cdot W(u)^2 \\
&= \int_{\mathbb{R}} y \cdot \frac{s \cdot y}{t} \cdot p_{W(t)}\big(y \mid W(u)\big)\,\mathrm{d}y - \frac{s \cdot t}{u^2} \cdot W(u)^2 \\
&= \frac{s}{t} \cdot \mathbb{E}\big[W(t)^2 \mid W(u)\big] - \frac{s \cdot t}{u^2} \cdot W(u)^2 \\
&= \frac{s}{t} \cdot \big(\mathrm{Var}[W(t) \mid W(u)] + \mathbb{E}[W(t) \mid W(u)]^2\big) - \frac{s \cdot t}{u^2} \cdot W(u)^2 \\
&= \frac{s}{t} \cdot \Big(\frac{t(u-t)}{u} + \frac{t^2}{u^2} \cdot W(u)^2\Big) - \frac{s \cdot t}{u^2} \cdot W(u)^2 \\
&= \frac{s(u-t)}{u},
\end{aligned}$$

where $p_{W(t)}(\cdot)$ is the probability density function of $W(t)$.
In summary, the Brownian bridge has a distribution with the above properties. Now it is natural to define the following *standard Brownian bridge* given these properties.

Definition 7 (Standard Brownian Bridge). A stochastic process $\{X(t)\}_{t \geq 0}$ is called a standard Brownian bridge, if $\{X(t)\}$ ends at $X(1) = 0$, that is,

1. $X(0) = X(1) = 0$;
2. $\mathrm{Cov}(X(s), X(t)) = s(1-t)$.

Remark. We now construct a standard Brownian bridge. Let

$$X(t) \triangleq W(t) - t \cdot W(1).$$

Then we claim that $\{X(t)\}$ is a standard Brownian bridge.

Example 8. [Hitting Probability in a Brownian Bridge] Recall that

$$\tau_b \triangleq \inf\{t \geq 0 : W(t) > b\}.$$

Now we are interested in $\mathbf{Pr}[\tau_b < u \mid W(u) = x]$, namely, the probability of hitting b before ending at $W(u) = x$. Clearly, if $b \leq x$, then $\tau_b \leq u$. So we assume that $b > x$.

Note that

$$\mathbf{Pr}[\tau_b < u \mid W(u) = x] = \frac{\mathbf{Pr}[\tau_b < u \wedge W(u) = x]}{\mathbf{Pr}[W(u) = x]}.$$

Informally, let $\mathrm{d}x$ denote the interval $[x, x+h]$ as h is an infinitesimal. So we are interested in

$$\frac{\mathbf{Pr}[\tau_b < u \wedge W(u) \in \mathrm{d}x]}{\mathbf{Pr}[W(u) \in \mathrm{d}x]}. \tag{1}$$

We now compute the denominator and numerator respectively. Since $W(u) \sim \mathcal{N}(0, u)$, we obtain that

$$\frac{1}{\sqrt{u}} \cdot W(u) \sim \mathcal{N}(0, 1).$$

Hence, the denominator is

$$\begin{aligned}
\mathbf{Pr}[W(u) \in \mathrm{d}x] &= \mathbf{Pr}\left[\frac{1}{\sqrt{u}} \cdot W(u) \in \frac{\mathrm{d}x}{\sqrt{u}}\right] \\
&= \Phi\left(\frac{\mathrm{d}x}{\sqrt{u}}\right) = \frac{1}{\sqrt{u}} \cdot \varphi\left(\frac{x}{\sqrt{u}}\right)\mathrm{d}x
\end{aligned} \tag{2}$$

where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-x^2/2}$$

is the probability density function of the standard normal distribution. Applying the reflection principle, the numerator is

$$\begin{aligned}
\mathbf{Pr}[\tau_b < u \wedge W(u) \in \mathrm{d}x] &= \mathbf{Pr}[\tau_b < u] \cdot \mathbf{Pr}[W(u) \in \mathrm{d}x \mid \tau_b < u] \\
&= \mathbf{Pr}[\tau_b < u] \cdot \mathbf{Pr}[W(u) \in 2b - \mathrm{d}x \mid \tau_b < u] \\
&= \mathbf{Pr}[\tau_b < u \wedge W(u) \in 2b - \mathrm{d}x] \\
&= \mathbf{Pr}[W(u) \in 2b - \mathrm{d}x] \\
&= \frac{1}{\sqrt{u}} \cdot \varphi\left(\frac{2b - x}{\sqrt{u}}\right)\mathrm{d}x.
\end{aligned} \tag{3}$$

Plugging (2) and (3) into (1) we conclude that

$$\mathbf{Pr}[\tau_b < u \mid W(u) = x] = \exp\left(-\frac{2b(b-x)}{u}\right).$$

Example 9 (Kolmogorov–Smirnov test). Suppose there is an oracle that claims to be able to generate random numbers. We would like to check if it is indeed a random number generator.

Suppose that $U_1, U_2, \ldots \in [0, 1]$ are sampled from distribution F (by the oracle) where F is the cumulative distribution function. Our goal is to determine whether $F(t) = t$ holds for all $0 < t < 1$. Let

$$F_n(t) = \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbb{1}_{[U_i < t]}.$$

Then the algorithm (Kolmogorov–Smirnov test) outputs

- reject, if $F(t) - t \geq b$ for some $t \in [0,1]$;
- accept, otherwise.

Now the problem is to determine the minimun value of b to make the probability of failure sufficiently small (i.e., $< 1/100$). We further assume that $F(t) = t$, and we hope the algorithm output accept.

Applying the central limit theorem, $F_n(t)$ has a normal distribution with mean t as n goes to infinity. For fixed t, we have

$$\mathsf{Var}\left[\mathbb{1}_{[U_i < t]}\right] = t(1-t)\,.$$

Now let

$$X_n(t) \triangleq \sqrt{n} \cdot (F_n(t) - t)\,.$$

By the central limit theorem, it follows that

$$X_n(t) \xrightarrow{D} \mathcal{N}\big(0, t(1-t)\big)\,.$$

In fact, using the high dimensional central limit theorem, we further obtain that

$$\big(X_n(t_1), X_n(t_2)\big)^\top \xrightarrow{D} \mathcal{N}\big(0, t_1(1-t_2)\big)$$

where $t_1(1-t_2)$ is the covariance of $X_n(t_1)$ and $X_n(t_2)$ since

$$\begin{aligned}\mathsf{Cov}\big(\mathbb{1}_{[U<t_1]}, \mathbb{1}_{[U<t_2]}\big) &= \mathbb{E}\left[\mathbb{1}_{[U<t_1]} \cdot \mathbb{1}_{[U<t_2]}\right] - t_1 \cdot t_2 \\ &= t_1 - t_1 \cdot t_2 \; = t_1(1-t_2)\,.\end{aligned}$$

Thus, it implies that

$$\big(X_n(t_1), X_n(t_2), \ldots, X_n(t_k)\big)^\top \xrightarrow{D} \big(W_n(t_1), W_n(t_2), \ldots, W_n(t_k)\big)^\top$$

where $W_n(t)$ is a standard Brownian bridge. Using the result in Example 8, we conclude that

$$\mathbf{Pr}[F_n(t) - t \geq b] = \exp\big(-2b^2\big)\,,$$

and hence we could obtain the minimun value of b by solving the inequality $\exp\big(-2b^2\big) < 1/100$.