Lec 3.

Divide-and-Conquer.

① Divide the problem into small instances

②. Solve small instances recursively

③. Combine them cleverly.

Ex. merge sort.

Comparison based $\Omega(n\log n)$.

Word Ram: Han & Thorup $O(n\sqrt{\log\log n})$.

Ex Counting inverse pairs.

Chan & Pǎtraşcu. $O(n\sqrt{\log n})$.

Master Theorem.

$$T(n) = a\,T(\lceil n/b \rceil) + O(n^d). \qquad a>0,\ b>1,\ d\geq 0$$

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & d = \log_b a \\ O(n^{\log_b a}) & d < \log_b a. \end{cases}$$

pf of MT.

wlog $n = b^k$ for $k \in \mathbb{N}$

$$\left.\begin{array}{ll} [n] & n^d \\ \downarrow & \\ a \times [n/b] & a \times (n/b)^d \\ a^2 \times [n/b^2] & a^2 \times (n/b^2)^d \\ \vdots & \end{array}\right\} k$$

Cost at i-th level.

$$n^d \cdot \left(\frac{a}{b^d}\right)^k$$

$$\Rightarrow T(n) = n^d \sum_{i=0}^{k} \left(\frac{a}{b^d}\right)^i$$

① $a < b^d$: n^d

② $a = b^d$: $n^d \log_b n$

③ $a > b^d$

$$n^d \cdot \left(\frac{a}{b^d}\right)^{\log_b n}$$

$$= \cancel{n^d} \cdot \frac{a^{\log_b n}}{\cancel{(b^{\log_b n})^d}} = n^{\log_b a}$$

Strassen's Algorithm

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

$$XY = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$T(n) = 8T(n/2) + O(n^2) = O(n^3).$$

$a = 8$, $b = 2$, $d = 2$

7 Multiplications suffices

$$XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}.$$

$P_1 = A(F-H)$. $\quad$ $P_5 = (A+D)(E+H)$

$P_2 = (A+B)H$ $\quad$ $P_6 = (B-D)(G+H)$

$P_3 = (C+D)E$ $\quad$ $P_7 = (A-C)(E+F)$.

$P_4 = D(G-E)$

$$T(n) = 7T(n/2) + O(n^2) = O(n^{\log_2 7}) = O(n^{2.81})$$

$O(n^w)$ $\quad 2 \le w < 2.373$

[Alman & Williams 2020].

Quick Select.

$X_1 = n$. The threshold x is ℓ-th largest element.

$$E[X_2 \mid X_1] \le E[\max\{\ell-1, n-\ell\}].$$

$$E[\max\{\ell-1, n-\ell\}] = E[\ell-1 \mid \ell-1 \ge n-\ell]\Pr[\ell-1 \ge n-\ell]$$
$$+ E[n-\ell \mid \ell-1 < n-\ell]\Pr[\ell-1 < n-\ell]$$

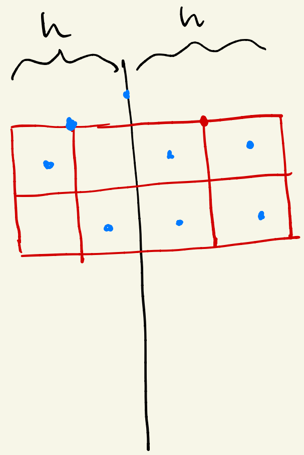
$$\le \frac{3}{4} n. = \frac{3}{4} X_1.$$

Closest pair in $\mathbb{R}^2$

sort in x_i.

* divide

* conquer.



$h = \min h_1, h_2$

$1 \times [n]$ $\quad c \cdot n$

$4 \times [\frac{n}{2}]$ $\quad c \cdot \frac{n}{2}$

$16 \times [\frac{n}{4}]$ $\quad c \cdot \frac{n}{4}$

$\vdots$

Multiplication.

$$x = 2^{n/2} x_L + x_R$$

$$y = 2^{n/2} y_L + y_R.$$

$$xy = 2^n x_L y_L + 2^{n/2}(x_L y_R + y_L x_R) + x_R y_R.$$

$$T(n) = 4T(n/2) + O(n) = O(n^2).$$

$$x_L x_R + y_L x_R = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$$

$$T(n) = 3T(n/2) + O(n) = O(n^{\log_2 3}) \approx O(n^{1.59}).$$

FFT $O^*(n \log n)$. $\quad \Omega(n)$.

[2019. Harvey & Van der Hoeven $O(n \log n)$].