[CS3958: Lecture 4] Optional Stopping Theorem

Instructor: Chihao Zhang, Scribed by Yulin Wang

January 4, 2023

1 Optional Stopping Theorem

First we review the statement of the optional stopping theorem and prove it.

Theorem 1 (Optional Stopping Theorem) *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if* at least one of *the following conditions holds:*

1. *τ is bounded almost surely, that is, $\exists n \in \mathbb{N}$ such that $\mathbf{Pr}[\tau \leq n] = 1$;*
2. *$\mathbf{Pr}[\tau < \infty] = 1$, and there is a finite M such that $|X_t| \leq M$ for all $t < \tau$;*
3. *$\mathbf{E}[\tau] < \infty$, and there is a constant c such that $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$ for all $t < \tau$.*

Proof. It is obvious that for every $n \in \mathbb{N}$, $\mathbf{E}[X_n] = \mathbf{E}[X_0]$. So first we show that for every $n \in \mathbb{N}$, $\mathbf{E}\left[X_{\min\{n,\tau\}}\right] = \mathbf{E}[X_0]$. Define $Z_n \triangleq X_{\min\{n,\tau\}} = X_0 + \sum_{i=0}^{n-1}(X_{i+1} - X_i)\mathbb{1}[\tau > i]$. We verify that $\{Z_n\}_{n\geq 0}$ is a martingale. By definition

$$\begin{aligned}
\mathbf{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbf{E}[Z_n + (X_{n+1} - X_n)\mathbb{1}[\tau > n] \mid \mathcal{F}_n] \\
&= Z_n + \mathbb{1}[\tau > n]\left(\mathbf{E}[X_{n+1} \mid \mathcal{F}_n] - X_n\right) \\
&= Z_n.
\end{aligned}$$

So we have $\mathbf{E}\left[X_{\min\{n,\tau\}}\right] = \mathbf{E}[Z_n] = \mathbf{E}[Z_0] = \mathbf{E}[X_0]$.

Therefore, this motivates us to decompose X_τ into two terms:

$$\forall n \in \mathbb{N},\ X_\tau = X_{\min\{n,\tau\}} + \mathbb{1}[\tau > n] \cdot (X_\tau - X_n).$$

Taking expectation and letting n tend to infinity, we obtain

$$\mathbf{E}[X_\tau] = \mathbf{E}[X_0] + \lim_{n\to\infty} \mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)].$$

Therefore, we only need to verify that each of the three conditions in the statement guarantee $\lim_{n\to\infty} \mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] = 0$.

1. If τ is bounded almost surely, then clearly $\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] = 0$ for sufficiently large n.
2. In this case,

$$\begin{aligned}
\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] &\leq \mathbf{E}[\mathbb{1}[\tau > n] \cdot (|X_\tau| + |X_n|)] \\
&\leq 2M \cdot \mathbf{E}[\mathbb{1}[\tau > n]] \\
&= 2M \cdot \mathbf{Pr}[\tau > n] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

3. In order to apply our bounds on the gap between consecutive X_t, we write

$$\begin{aligned}\mathbb{1}[\tau > n] \cdot (X_\tau - X_n) &= \sum_{t=n}^{\tau-1} (X_{t+1} - X_t) \\ &\le \sum_{t=n}^{\tau-1} |X_{t+1} - X_t| \\ &= \sum_{t=n}^{\infty} |X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t].\end{aligned}$$

Taking expectation on both sides, we have

$$\begin{aligned}\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] &\le \mathbf{E}\left[\sum_{t=n}^{\infty} |X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t]\right] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[|X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t]] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[\mathbf{E}[|X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t] \mid \mathcal{F}_t]] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \cdot \mathbb{1}[\tau > t]] \\ &\le \sum_{t=n}^{\infty} c \cdot \mathbf{Pr}[\tau > t],\end{aligned}$$

where the first equality follows from the monotone convergence theorem.

On the other hand, we know $\mathbf{E}[\tau] = \sum_{t=0}^{\infty} \mathbf{Pr}[\tau > t] < \infty$. Therefore, the tail of this sequence, $\sum_{t=n}^{\infty} \mathbf{Pr}[\tau > t] \to 0$ as $n \to \infty$.

□

2 Applications of OST

2.1 Doobs martingale inequality

With OST, we can obtain concentration property of the maximum element in a seuqence of random variables.

Claim 2 *Let $\{X_t\}_{t\ge 0}$ be a martingale with respect to itself where $X_t \ge 0$ for every t. Prove that for every $n \in \mathbb{N}$,*

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] \le \frac{\mathbf{E}[X_0]}{\alpha}.$$

Proof. We define a stopping time τ when the first element that is greater that α occurs, and otherwise set $\tau = n$. Formally, define

$$\tau \triangleq \min\left(n, \min_{t\le n}\{t \mid X_t \ge \alpha\}\right).$$

By definition of τ, we have

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] = \mathbf{Pr}\left[X_\tau \ge \alpha\right].$$

Since τ is bounded, we apply Optional Stopping Theorem to obtain that $E[X_\tau] = E[X_0]$. Therefore, by Markov's Inequality,

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] = \mathbf{Pr}\left[X_\tau \ge \alpha\right] \le \frac{\mathbf{E}\left[X_\tau\right]}{\alpha} = \frac{\mathbf{E}\left[X_0\right]}{\alpha}$$

□

2.2 *One-dimensional Random Walk with Two Absorbing Barriers*

We've discussed one-dimensional random walk with one absorbing barrier before

We consider another problem in one-dimensional random walk. Let $a, b > 0$ be two integers. A man starts the random walk from 0 and stops when he arrives at $-a$ or b. Let τ be the time when the man first reaches $-a$ or b, i.e., the first time t that $X_t = -a$ or $X_t = b$. The model is called "one-dimensional random walk with two absorbing barriers". We want to compute the expected value of $\mathbf{E}[\tau]$, that is, the average stopping time of the walk.

We want to construct a martingale $\{Y_t\}_{t\ge0}$ such that OST can be applied to $\{Y_t\}_{t\ge0}$ and τ and thereby we can derive an equality related to $\mathbf{E}[\tau]$. Before calculating $\mathbf{E}[\tau]$, we first determine $\mathbf{Pr}[X_\tau = -a]$, the probability that the man stops at position $-a$. Let $P_a \triangleq \mathbf{Pr}[X_\tau = -a]$. We want to apply OST to show $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$. Therefore, we verify that some of conditions in OST is satisfied.

In a time period of length $T = a + b$, if the man walks towards the same direction, he must have stopped, either at $-a$ or b, which happens with probability $2^{-(a+b)}$. Therefore, if we divide the time into consecutive periods in this manner, in expected finite time, we can meet some period when the event happened. Hence, $\mathbf{E}[\tau] < \infty$. Moreover, we clearly have $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] < 1$ for every $0 \le t < \tau$, so the third condition of OST holds, which implies that $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$. On the other hand, we have $\mathbf{E}[X_\tau] = P_a \cdot (-a) + (1 - P_a) \cdot b$. These two equalities give $P_a = \frac{b}{a+b}$.

Then for all $t \ge 0$, we define a new random variable $Y_t \triangleq X_t^2 - t$ which involves the time t. The following fact is easy to verify by definition.

Claim 3 *$\{Y_t\}_{t\ge0}$ is a martingale.*

Proof. First we have

$$\begin{aligned}\mathbf{E}\left[Y_{t+1} \mid \mathcal{F}_t\right] &= \mathbf{E}\left[X_{t+1}^2 - (t+1) \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbf{E}\left[(X_t + c_t)^2 - (t+1) \,\middle|\, \mathcal{F}_t\right] \\ &= \mathbf{E}\left[X_t^2 \,\middle|\, \mathcal{F}_t\right] + 2\mathbf{E}\left[X_t c_t \mid \mathcal{F}_t\right] + \mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] - (t+1).\end{aligned}$$

Since X_t is $\mathcal{F}_t$-measurable, $\mathbf{E}[c_t \mid \mathcal{F}_t] = 0$ and $\mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] = 1$, we can further derive that

$$\mathbf{E}\left[Y_{t+1} \mid \mathcal{F}_t\right] = X_t^2 + 0 + 1 - (t+1) = X_t^2 - t = Y_t.$$

Hence $\{Y_t\}_{t\geq 0}$ is a martingale. □

Note that $X_t \in [-a, b]$ for all $t \geq 0$. Thus $|Y_{t+1} - Y_t| = \left|X_{t+1}^2 - (t+1) - X_t^2 + t\right| = \left|X_{t+1}^2 - X_t^2 - 1\right|$ is bounded by some constant. We can apply OST again to obtain $\mathbf{E}[Y_\tau] = \mathbf{E}[Y_0] = 0$. On the other hand, we have $\mathbf{E}[Y_\tau] = \mathbf{E}\left[X_\tau^2\right] - \mathbf{E}[\tau]$ by definition, and thus

$$\mathbf{E}[\tau] = \mathbf{E}\left[X_\tau^2\right] = a^2 P_a + b^2(1 - P_a) = a^2 \cdot \frac{b}{a+b} + b^2 \cdot \frac{a}{a+b} = ab.$$

Sometimes one can use OST in a reverse way. Consider the random walk with only one barrier at $-a$. The fact that $\mathbf{E}[\tau] = \infty$ can be proved in the following way (due to Biaoshuai Tao): If $\mathbf{E}[\tau] < \infty$, then by (cond 3 of) OST, $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$. On the other hand, we know $X_\tau = -a \neq 0$. Therefore it must be that $\mathbf{E}[\tau] = \infty$.

2.3 *Pattern Matching*

Suppose that there is a $\{H, T\}$-string P of length ℓ (H for "head" and T for "tail"). We flip a coin consecutively until the last ℓ results form exactly the same string as P. How many times do we flip the coin?

Note that if we flip the coin N times and observe the string S consisting of N results. No matter which pattern we choose, by the linearity of expectation, the expected number of occurrence [1] is

$$\mathbf{E}[\text{\# of occurrence of } P \text{ in } S] = \sum_{i=1}^{n-\ell+1} \mathbf{E}\left[\mathbb{1}[S_{i,i+1,\ldots,i+\ell-1} = P]\right] = (n - l + 1)\cdot 2^{-\ell}.$$

[1] That means the expected number of substrings exactly the same as P in the resulting string S.

However, if we would like to compute the first time that pattern P occurs, the pattern itself has an impact on the expected time. Intuitively, lets consider two patterns HT and HH. Assume that the first flipping result is H. Then we consider what happens if the second result fails. Suppose that the desired pattern is HT and H appears. Although we fail, we obtain an H. However, if the desired pattern is HH and the second flipping result is T, then we obtain nothing and the first two flips are a waste. So we should believe that the expected times of the first occurrence of HT is smaller than HH.

We now use the optional stopping theorem to solve this problem. Let $P = p_1 p_2 \ldots p_\ell$. For every $n \geq 0$, assume that before $n+1$-th flipping there is a new gambler G_{n+1} coming with 1 unit of money to bet that the following ℓ result (i.e., the $n+1$-th to $n+\ell$-th results) are exactly the same as P. At the $n+k$-th flipping, G_{n+1} will bet that the result is p_k by an all in strategy, that is, if the $n+k$-th result is p_k then G_{n+1} will have twice as much money as before; otherwise they will lose all. Suppose that the patter P = HTHTH and the flipping results are HTHHTHTH. The following table shows the total money of each gambler after flipping.

Let X_t be the result of t-th flipping, $M_i(t)$ denote the money that G_i has after t-th flipping, and $Z_t \triangleq \sum_{i=1}^t (M_i(t) - 1)$ be the total income of all gamblers after t-th flipping. It is easy to verify that $\{M_i(t)\}_{t\geq 0}$ is a martingale with respect to $\{X_t\}$ since

$$\mathbf{E}\left[M_i(t+1) \mid \bar{X}_{0,t}\right] = \frac{1}{2} \cdot 2M_i(t) + \frac{1}{2} \cdot 0 = M_i(t).$$

Then by the linearity of expectation we conclude that $\{Z_t\}_{t\geq 0}$ is a martingale with respect to the flipping results $\{X_t\}$ since $\mathbf{E}[M_i(t)] = 1$. Let

Gambler	H	T	H	H	T	H	T	H	Money	
1	H	T	H	T					0	1⟶2⟶4⟶8⟶0
2		H							0	1⟶0
3			H	T					0	1⟶2⟶0
4				H	T	H	T	H	32	1⟶2⟶4⟶8⟶16⟶32
5					H				0	1⟶0
2						H	T	H	8	1⟶2⟶4⟶8
5							H		0	1⟶0
5								H	2	1⟶2

τ be the stopping time defined by the first time that some gambler wins, namely, the first time that P occurs in the flipping results. Applying Condition 2 of OST we obtain that $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0] = 0$. Sequentially we have $\mathbf{E}\left[\sum_{i=1}^{\tau} M_i(\tau) - \tau\right] = 0$ and $\mathbf{E}[\tau] = \sum_{i=1}^{\tau} \mathbf{E}[M_i(\tau)]$.

Note that $M_i(t) = 0$ for $i \le \tau - \ell$ and $M_i(t) = 2^{\tau-i+1}\chi_{\tau-i+1}$ for $i > \tau - \ell$ where χ_j is defined by

$$\chi_j = \mathbb{1}[p_1 p_2 \dots p_j = p_{\ell-j+1} \dots p_{\ell-1} p_\ell].$$

Hence,

$$\mathbf{E}[\tau] = \sum_{i=\tau-\ell+1}^{\tau} \mathbf{E}[M_i(\tau)] = \sum_{i=1}^{\ell} 2^i \chi_i.$$

Recall the example of HH and HT. If P is HH, $\mathbf{E}[\tau] = 2 + 4 = 6$. If P is HT, $\mathbf{E}[\tau] = 4$. This confirms our hypothesis that $\mathbf{E}[\tau]$ for HH is larger than $\mathbf{E}[\tau]$ for HT.

2.4 Wald's Equation

In practice, we often need to analyze the (expected) running time of following procedure where both *cond* and *compute()* are random.

```
while cond do
    compute();
end while
```

Assume the i-th call to *compute()* costs X_i time and the algorithm terminates after T iterations. Then the total running time is $N \triangleq \sum_{i=1}^{T} X_i$. Suppose X_is are independently and identically distributed as a random variable X. The Wald's equation gives a formula for $\mathbf{E}[N]$.

Theorem 4 (Wald's Equation) *If we have*

- $X_1, X_2, \dots$ *are non-negative, independent, identically distributed random variables with the same distribution as* X.
- T *is a stopping time for* $X_1, X_2, \dots$.

- $\mathbf{E}[T], \mathbf{E}[X] < \infty$,

then

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X].$$

Proof. For $i \geq 1$, let $Z_i := \sum_{j=1}^{i}(X_j - \mathbf{E}[X])$. Clearly the sequence $Z_i, Z_2, \ldots$ is a martingale with respect to $X_1, X_2, \ldots$ and $\mathbf{E}[Z_1] = 0$. And we have

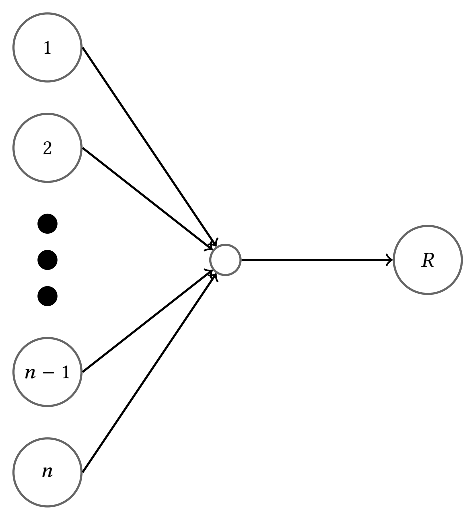
$$\begin{aligned}
\mathbf{E}[|Z_{i+1} - Z_i| \mid \mathcal{F}_i] &= \mathbf{E}[|X_{i+1} - \mathbf{E}[X]| \mid \mathcal{F}_i] \\
&\leq \mathbf{E}[X_{i+1} + \mathbf{E}[X] \mid \mathcal{F}_i] \\
&\leq 2\mathbf{E}[X].
\end{aligned}$$

We know that $\mathbf{E}[T], \mathbf{E}[X] < \infty$, and therefore applying OST derives $\mathbf{E}[Z_T] = \mathbf{E}[Z_1] = 0$. Then

$$\begin{aligned}
\mathbf{E}[Z_t] &= \mathbf{E}\left[\sum_{j=1}^{T}(X_j - \mathbf{E}[X])\right] \\
&= \mathbf{E}\left[\sum_{i=1}^{T} X_i - T\mathbf{E}[X]\right] \\
&= \mathbf{E}\left[\sum_{i=1}^{T} X_i\right] - \mathbf{E}[T]\,\mathbf{E}[X] = 0.
\end{aligned}$$

□

An Application of Wald's Equation: A Routing Problem



Let us consider an application of Wald's equation. There are n senders and one receiver. In each round, each sender sends a packet to the receiver with probability $\frac{1}{n}$. Since all senders share the same channel, if there are multiple packets sent at the same time, all of them will fail. The question is, on average, how many rounds are required so that each sender can successfully send at least one packet.

We let X_i be the variable indicating how long the receiver needs to get another packet after he has received $i - 1$ ones (counting packets from repeated sender). And let T be the number of packets received when first time the receiver receives at least one packet from each sender. The quantity we are interested in is

$$N \triangleq \sum_{i=1}^{T} X_i.$$

Clearly $X_1, X_2, \ldots$ are independently and identically distributed, and $\mathbf{E}[T]$ is finite. Therefore $\mathbf{E}[N] = \mathbf{E}[T] \cdot \mathbf{E}[X_1]$ by Wald's equation.

Note that by the definition, T is the number of coupons in the coupon collector's problem we met before. So $\mathbf{E}[T] = nH_n = \Theta(n \log n)$.

On the otherhand, $X_1 \sim \texttt{Geom}(p)$ with

$$p = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \approx e^{-1}$$

which implies $E[X_1] = e$. Therefore,

$$\mathbf{E}\left[N\right] = \mathbf{E}\left[T\right] \cdot \mathbf{E}\left[X_1\right] \approx enH_n.$$