Advanced Algorithms (VII)

Shanghai Jiao Tong University

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The Probabilistic Method

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CS477 Combinatorics (Spring 2019)

11. Lecture 11, 2019/05/17. Ramsey problem on integers. Erdos' proof of Ramsey lower bound.
12. Lecture 12, 2019/05/26. The probabilistic method.
13. Lecture 13, 2019/06/14. Erdos' theorem on large chromatic numbers and large girth. Set families. Some combin

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This is an important technique to prove the existence of some object.

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This is an important technique to prove the existence of some object.

Sometimes, it is also useful to “find the object”

Max Cut

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Given an undirected graph $G = (V, E)$, the max cut of G is the partition $V = S \cup \bar{S}$ such that $|E(S, \bar{S})|$ is maximized

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$\boxed{\text{edge between } S \text{ and } \bar{S}}$

It is NP-hard to determine the max cut exactly

On the other hand, each graph contains a cut of size at least $\dfrac{|E|}{2}$

We find a partition $(S, \bar{S})$ by tossing a fair coin at each vertex v

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So there exists a cut of size at least $\frac{|E|}{2}$

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The following straightforward strategy turns the argument into a *Las-Vegas algorithms*

"Repeat tossing coins until $|E(S, \bar{S})| \geq \dfrac{|E|}{2}$*"*

We know $\mathbf{E}[\,|E(S, \bar{S})|\,] = \dfrac{|E|}{2}$, so what is the expected running time of the algorithm?

Let p be the probability that our algorithm terminates in one round

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$$\frac{m}{2} = \mathbf{E}[|E(S, \bar{S})|] = \sum_{i=0}^{m} i \cdot \Pr[|E(S, \bar{S})| = i]$$

$$\leq \left(\frac{m}{2} - 1 \right)(1 - p) + pm$$

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So $p \geq \dfrac{2}{m+2}$

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Approximation Ratio of an algorithm A

for maximization problem:

$$\alpha(A) = \min_{G} \frac{A(G)}{\mathrm{OPT}(G)}$$

for minimization problem:

$$\alpha(A) = \max_{G} \frac{A(G)}{\mathrm{OPT}(G)}$$

Derandomization

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Let the coins be $X_1, X_2, \ldots, X_n$

We will decompose $\mathbf{E}[\,|E(S, \bar{S})|\,]$ using conditional expectation

$$
\begin{aligned}
\mathbf{E}[\,|E(S,\bar{S})|\,] &= \mathbf{E}[\mathbf{E}[\,|E(S,\bar{S})|\,|X_1, X_2, \ldots, X_n]] \\
&= \frac{1}{2}\cdot \mathbf{E}[\mathbf{E}[\,|E(S,\bar{S})|\mid X_1 = 0, X_2, \ldots, X_n]] \\
&\quad + \frac{1}{2}\cdot \mathbf{E}[\mathbf{E}[\,|E(S,\bar{S})|\mid X_1 = 1, X_2, \ldots, X_n]] \\
&= \frac{1}{2}\cdot \mathbf{E}[\,|E(S,\bar{S})|\mid X_1 = 0]] + \frac{1}{2}\cdot \mathbf{E}[\,|E(S,\bar{S})|\mid X_1 = 1]]
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So we know at least one of $E_0 \geq \dfrac{m}{2}$ and $E_1 \geq \dfrac{m}{2}$ holds

Moreover, both E_0 and E_1 can be efficiently computed

We can set $X_1 = 0$ or $X_1 = 1$ according to which of E_0 and E_1 is bigger

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We obtained the approximation ratio of the greedy algorithm as a byproduct

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Input: A CNF formula $\phi = C_1 \wedge C_2 \cdots \wedge C_m$

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Formula ϕ, variables $V = \{x_1, \ldots, x_n\}$, $|C_i| = \ell_i \geq 1$

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$$\mathbf{E}[X] = \sum_{i=1}^{m} \Pr[C_i \text{ is satisfied}] = \sum_{i=1}^{m} \left(1 - 2^{-\ell_i}\right) \geq \frac{m}{2}$$

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To bound the approximation ratio, we need an upper bound for $\mathrm{OPT}(\phi)$

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In fact, we can tweak those singleton clauses

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Both cases are good for us!

Assume there are more positive singletons than negative singletons in ϕ

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Switch the positive and the negative for all appearance of x

$$\mathbf{E}[X] = t + \sum_{C \in \mathscr{C}'} \Pr[C \text{ is satisfied}] \geq t + (m - 2t) \min\{p, 1 - p^2\}$$

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For $p = 1 - p^2$, we have a 0.618-approximation algorithm

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We first treat MaxSAT problem as an *integer programming*

The Integer Program

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\begin{aligned}
\max \quad & \sum_{j=1}^{m} z_j \\
\text{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{k \in N_j} (1 - y_k) \geq z_j, \quad \forall j \in [m] \text{ s.t. } C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{k \in N_j} \bar{x}_k \\
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$$\mathrm{OPT}(\phi) \leq \mathrm{OPT}(LP) = \sum_{j=1}^{m} z_j^*$$

$$
\begin{aligned}
\Pr[C_j \text{ is not satisfied}] &= \prod_{i \in P_j} (1 - y_i^*) \prod_{k \in N_j} y_k^* \\
&\leq \left(\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{k \in N_j} y_k^* \right) \right)^{\ell_j} \\
&= \left(\frac{1}{\ell_j} \left(\ell_j - \left(\sum_{i \in P_j} y_i^* + \sum_{k \in N_j} (1 - y_k^*) \right) \right) \right)^{\ell_j} \\
&\leq \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j}.
\end{aligned}
$$

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\Pr[C_j \text{ is not satisfied}] &= \prod_{i \in P_j} (1 - y_i^*) \prod_{k \in N_j} y_k^* \\
\boxed{\text{AM-GM}} \qquad &\leq \left(\frac{1}{\ell_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{k \in N_j} y_k^* \right) \right)^{\ell_j} \\
&= \left(\frac{1}{\ell_j} \left(\ell_j - \left(\sum_{i \in P_j} y_i^* + \sum_{k \in N_j} (1 - y_k^*) \right) \right) \right)^{\ell_j} \\
&\leq \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{j=1}^{m} \Pr[C_j \text{ is satisfied}] \\
&\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{z_j^*}{\ell_j} \right)^{\ell_j} \right) \\
&\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{1}{\ell_j} \right)^{\ell_j} \right) z_j^* \\
&\geq (1 - e^{-1}) \sum_{j=1}^{m} z_j^* \geq \left(1 - \frac{1}{e} \right) \mathrm{OPT}
\end{aligned}
$$

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\boxed{\text{Concavity}} \quad &\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{1}{\ell_j} \right)^{\ell_j} \right) z_j^* \\
&\geq (1 - e^{-1}) \sum_{j=1}^{m} z_j^* \geq \left(1 - \frac{1}{e} \right) \mathrm{OPT}
\end{aligned}
$$

$$\mathbf{E}[X] = \sum_{j=1}^{m} \Pr[C_j \text{ is satisfied}]$$

$$\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{z_j^*}{\ell_j}\right)^{\ell_j}\right)$$

Concavity

$$\geq \sum_{j=1}^{m} \left(1 - \left(1 - \frac{1}{\ell_j}\right)^{\ell_j}\right) z_j^*$$

$$\geq (1 - e^{-1}) \sum_{j=1}^{m} z_j^* \geq \left(1 - \frac{1}{e}\right) \text{OPT}$$

≈ 0.632