Advanced Algorithms (VI)

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A martingale is a sequence of pairs $\{X_t, \mathscr{F}_t\}_{t\geq 0}$ s.t.

- for all $t \geq 0$, X_t is $\mathscr{F}_t$-measurable
- for all $t \geq 0$, $\mathbf{E}[X_{t+1} \mid \mathscr{F}_t] = X_t$

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“whether to stop can be determined by looking at the outcomes seen so far”

- The first time a gambler wins five games in a row
- The last time a gambler wins five games in a row

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Not true in general. Assume τ is the first time a gambler wins \$100

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- $\Pr[\tau < \infty] = 1$
- $\mathbf{E}[\,|X_\tau|\,] < \infty$
- $\lim_{t\to\infty} \mathbf{E}[X_t \cdot \mathbf{1}_{[\tau>t]}] = 0$

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OST applies when at least one of above holds

Proof of the Optional Stopping Theorem

Applications of OST

Random Walk in 1-D

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What is $\mathbf{E}[\tau]$?

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$$\implies p_a = \frac{b}{a+b}$$

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$$\begin{aligned}
\mathbf{E}[Y_{t+1} \mid \mathcal{F}_t] &= \mathbf{E}[(X_t + Z_{t+1})^2 - (t+1) \mid \mathcal{F}_t] \\
&= \mathbf{E}[X_t^2 + 2Z_{t+1}X_t - t \mid \mathcal{F}_t] \\
&= X_t^2 - t = Y_t
\end{aligned}$$

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This implies $\mathbf{E}[\tau] = ab$

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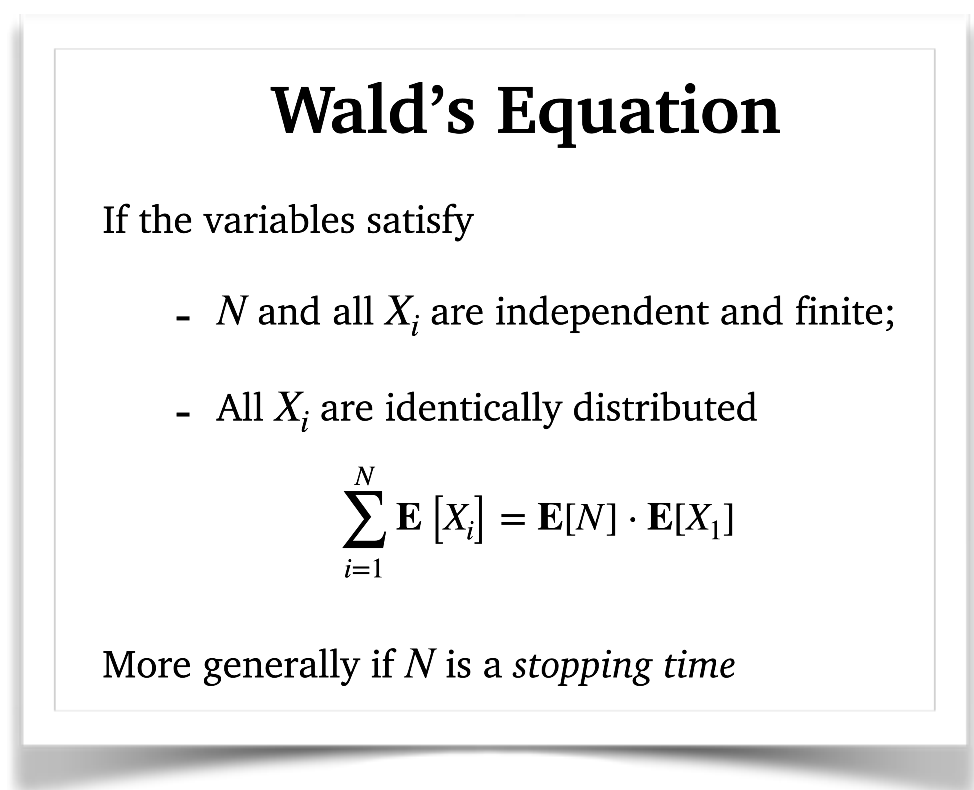

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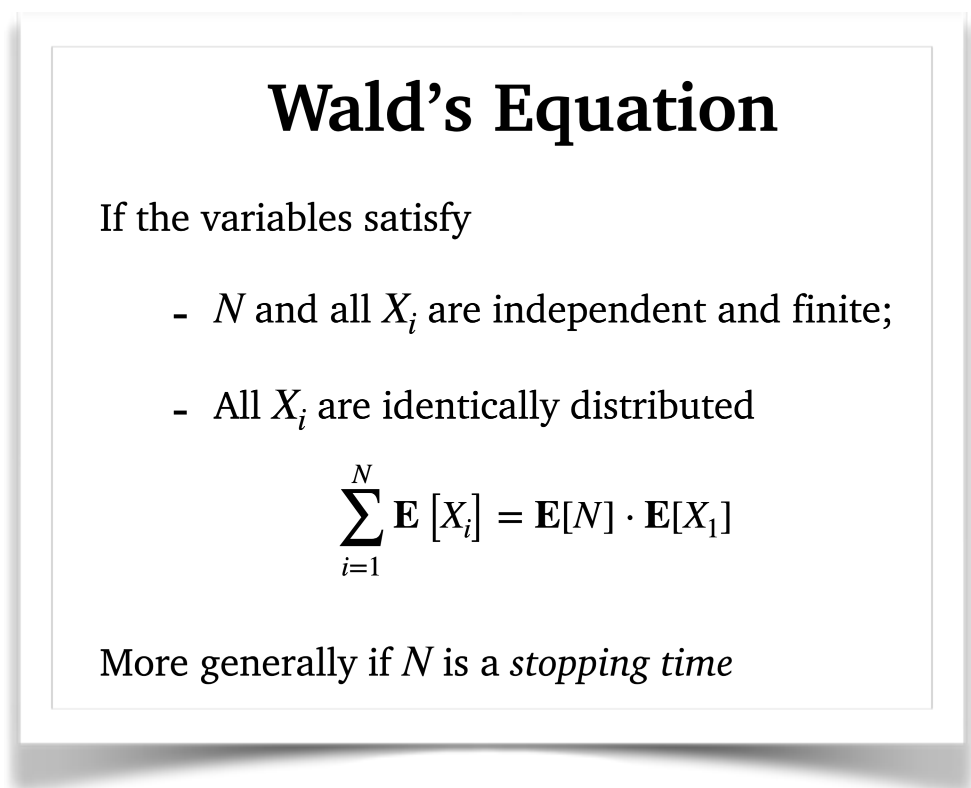
We are now ready to prove the general case!



Wald's Equation

If the variables satisfy

- N and all X_i are independent and finite;
- All X_i are identically distributed

$$\sum_{i=1}^{N} \mathbf{E}\left[X_i\right] = \mathbf{E}[N] \cdot \mathbf{E}[X_1]$$

More generally if N is a *stopping time*

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$$\mathbf{E}[Y_N] = \mathbf{E}\left[\sum_{i=1}^{N} (X_i - \mu)\right] = \mathbf{E}\left[\sum_{i=1}^{N} X_i\right] - \mathbf{E}\left[\sum_{i=1}^{N} \mu\right]$$

$$= \mathbf{E}\left[\sum_{i=1}^{N} X_i\right] - \mathbf{E}[N] \cdot \mu = 0$$

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Shuo-Yen Robert Li (李碩彥)

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At time j, G_j bets \$1 for "$b_j = p_1$". If he wins, he bets \$2 for "$b_{j+1} = p_2$", ...

He keeps doubling the money until he loses

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$\{X_t\}$ and τ meet the conditions for OST, so $\mathbf{E}[X_\tau] = 0$

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If $G_{\tau-j+1}$ wins, he wins $\$2^j - 1$

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contribution of losers

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$$\text{This implies } \mathbf{E}[\tau] = \sum_{j=1}^{k} \chi_j \cdot 2^j$$

Proof of OST

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Show on Board

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Read Chapter 8 of "Notes on Randomized Algorithms" for more details

https://arxiv.org/abs/2003.01902