Advanced Algorithms (XIII)

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Total Variation Distance

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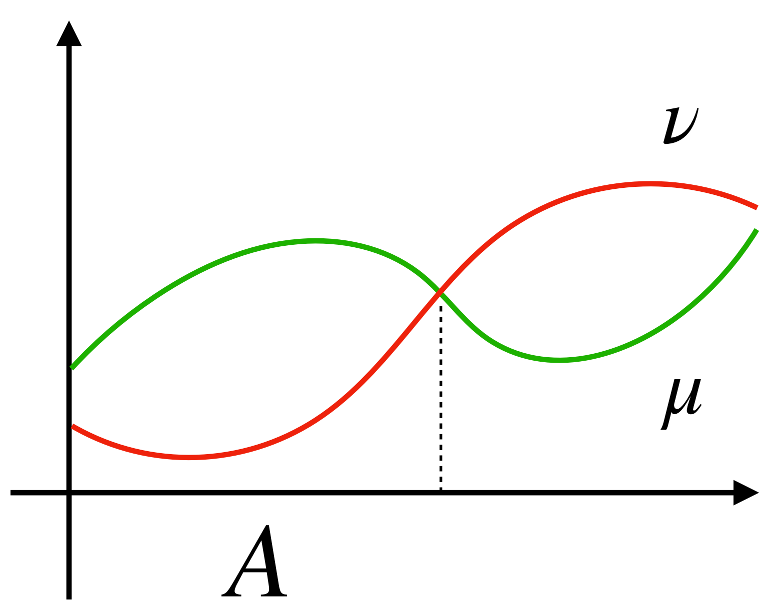

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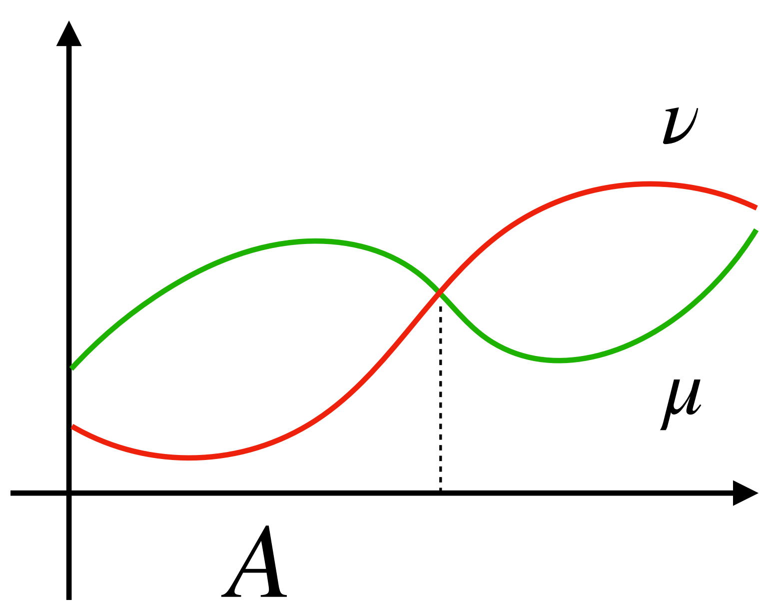

ℓ_1-distance scaled by $\frac{1}{2}$

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ω^* is the one maximizing the sum of diagonals

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A coupling of the two chains is joint distribution ω of $\{\mu_t\}_{t\geq 0}$ and $\{\nu_t\}_{t\geq 0}$ satisfying the following conditions

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Two chains coalesce once they meet

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$$\lim_{n \to \infty} \mathbf{Pr}[X_n \neq Y_n] = 0$$

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$$\tau_{\text{mix}} = \tau_{\text{mix}}(1/4)$$

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Lazy walk on G

Standing at $x \in \{0,1\}^n$

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- otherwise, choose $i \in [n]$ u.a.r and flip $x(i)$

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We couple them by choosing the same i and b

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$$\tau_{\text{mix}}(\varepsilon) \leq n \log n + n \log \varepsilon^{-1}$$

Another Random Walk

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A coupling argument implies $\tau_{\text{mix}} \leq \frac{1}{2} n \log n + O(n)$

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We showed that spectral decomposition is a powerful tool to analyze reversible chains

Relaxation Time

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$$\boxed{P = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T D_\pi}$$

$$P^t = \sum_{i=1}^{n} \lambda_i^t \mathbf{v}_i \mathbf{v}_i^T D_\pi$$

- $\lambda_n = 1$
- $\lambda_1 \geq -1$ and $\lambda_1 = -1$ if and only if P is bipartite
- $\lambda_{n-1} = 1$ if and only P is reducible

If P is reducible ($\lambda_{n-1} < 1$) and aperiodic ($\lambda_1 > -1$)

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For reversible, irreducible, aperiodic chains:

$$(\tau_{\text{rel}} - 1)\log\left(\frac{1}{2\varepsilon}\right) \leq \tau_{\text{mix}}(\varepsilon) \leq \tau_{\text{rel}} \log\left(\frac{1}{\varepsilon \pi_{\min}}\right)$$

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$$\lim_{t\to\infty} P^t = \mathbf{1}\mathbf{1}^T D_\pi = \begin{bmatrix} \pi^T \\ \pi^T \\ \vdots \\ \pi^T \end{bmatrix}$$

$$\lambda_\star := \max_{1\le i\le n-1} |\lambda_i|$$

$$\tau_{\text{rel}} := \frac{1}{1-\lambda_\star}$$

For reversible, irreducible, aperiodic chains:

$$\pi_{\min} := \min_x \pi(x)$$

$$(\tau_{\text{rel}} - 1)\log\left(\frac{1}{2\varepsilon}\right) \le \tau_{\text{mix}}(\varepsilon) \le \tau_{\text{rel}} \log\left(\frac{1}{\varepsilon\pi_{\min}}\right)$$