Advanced Algorithms (XII)

Shanghai Jiao Tong University

Chihao Zhang

May 25, 2020

Random Walk on a Graph

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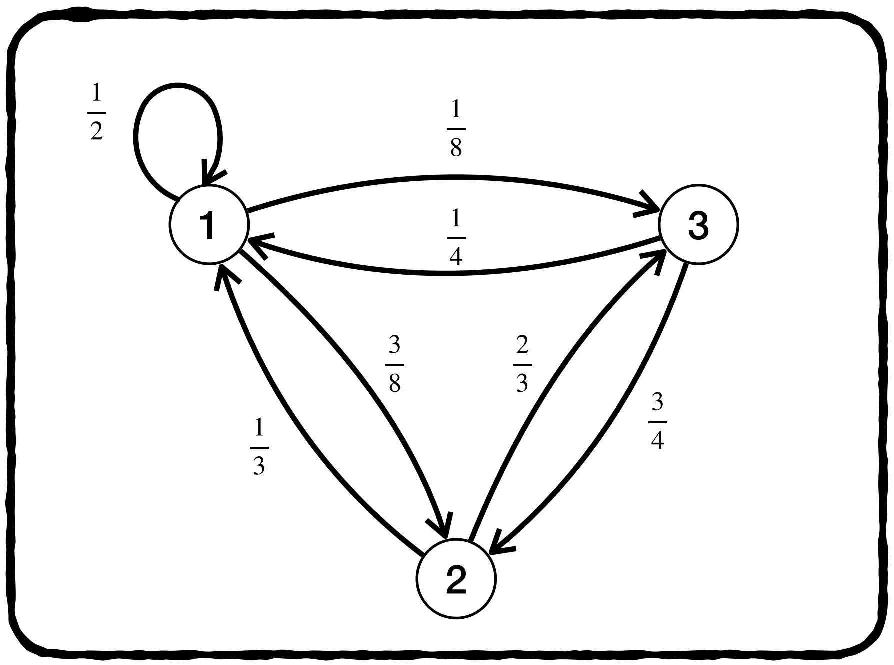

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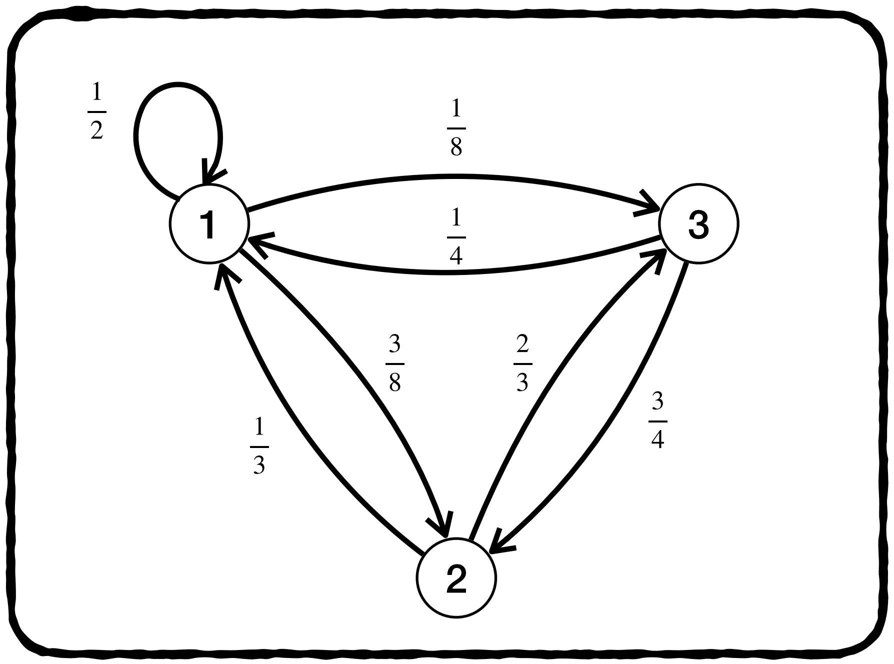

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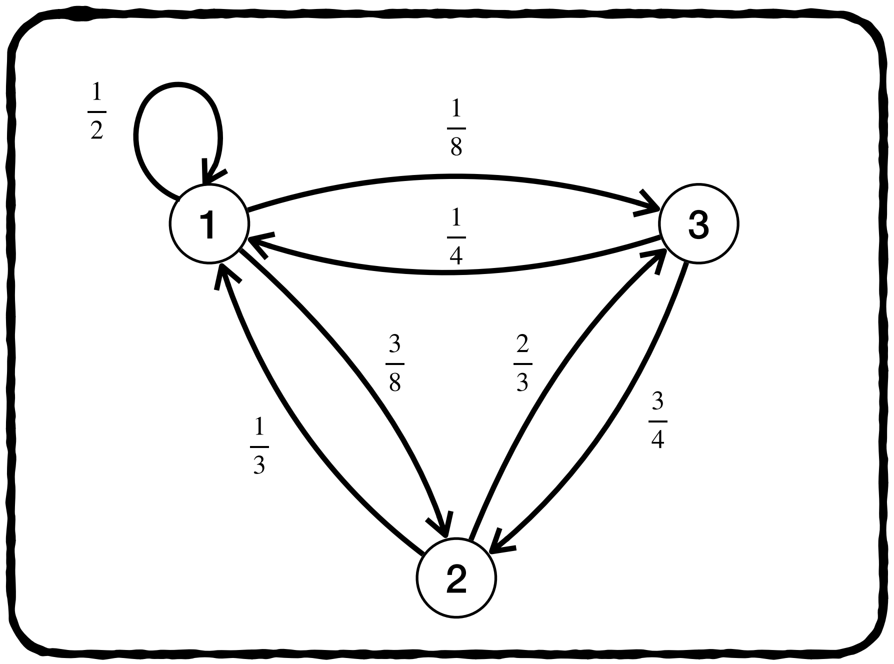

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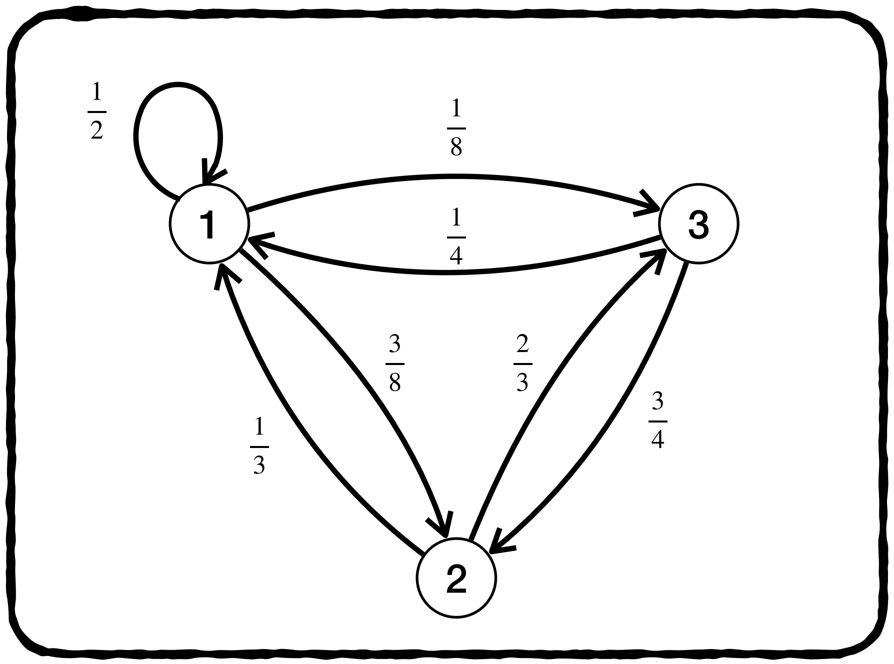

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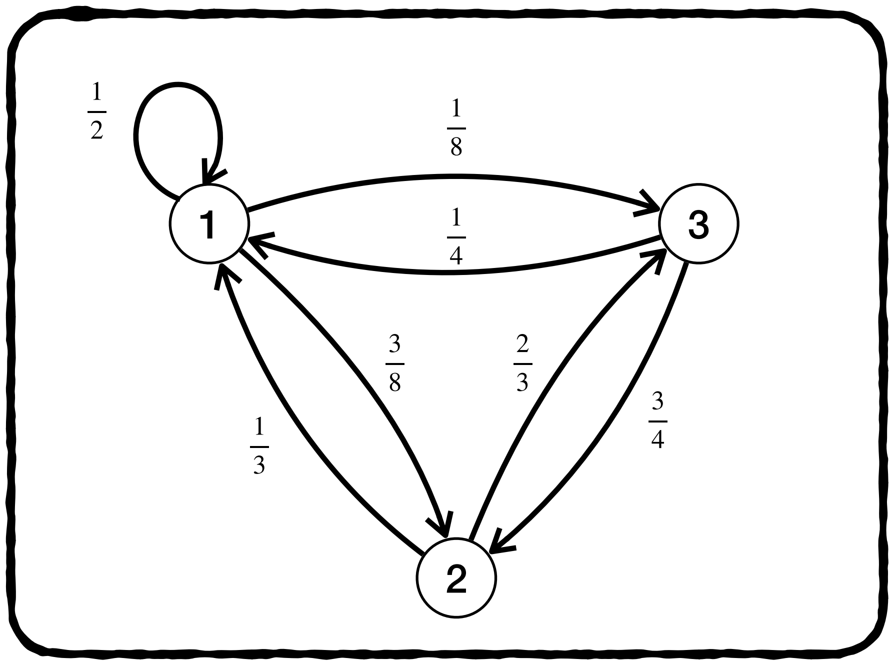

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Stationary distribution π: $\pi^T P = \pi^T$

Fundamental Theorem of Markov Chains

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- If so, does any initial distribution converge to it?

Existence of Stationary Distribution

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The positive eigenvector is π

Uniqueness and Convergence

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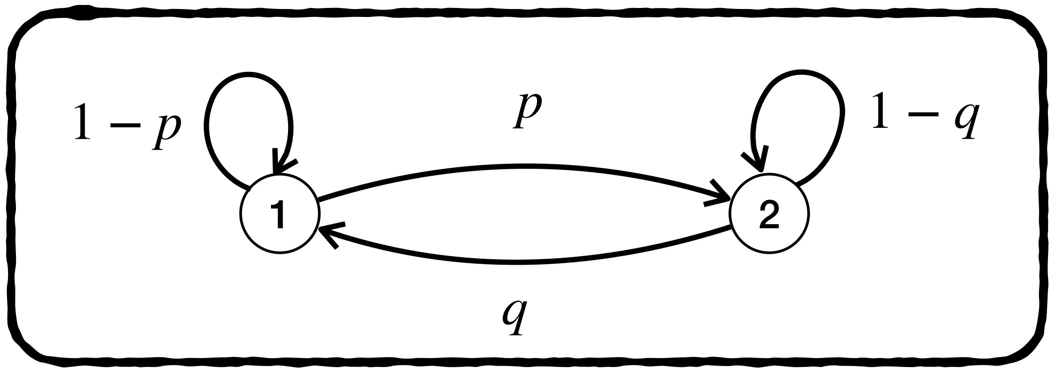

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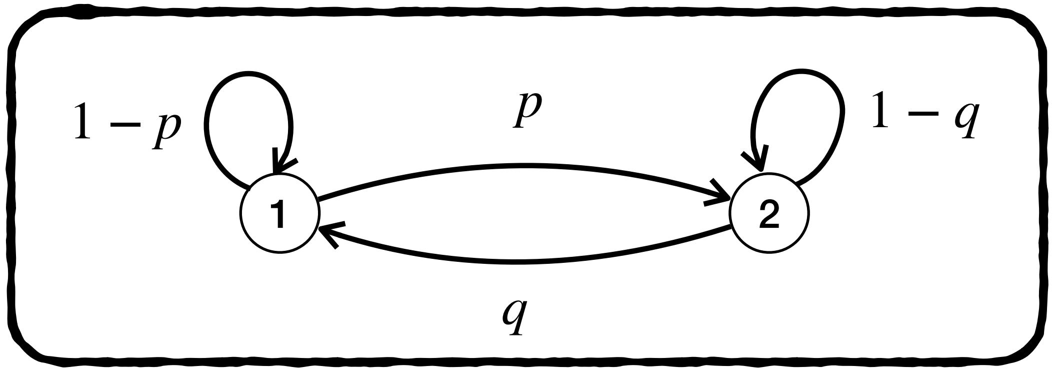

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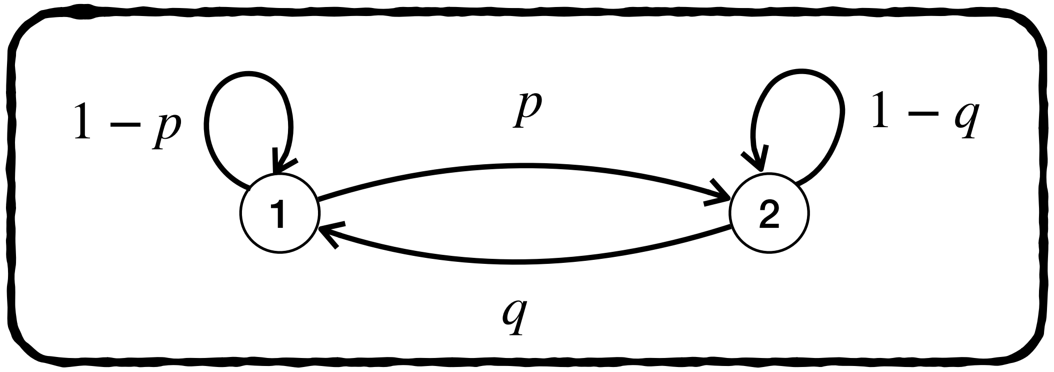

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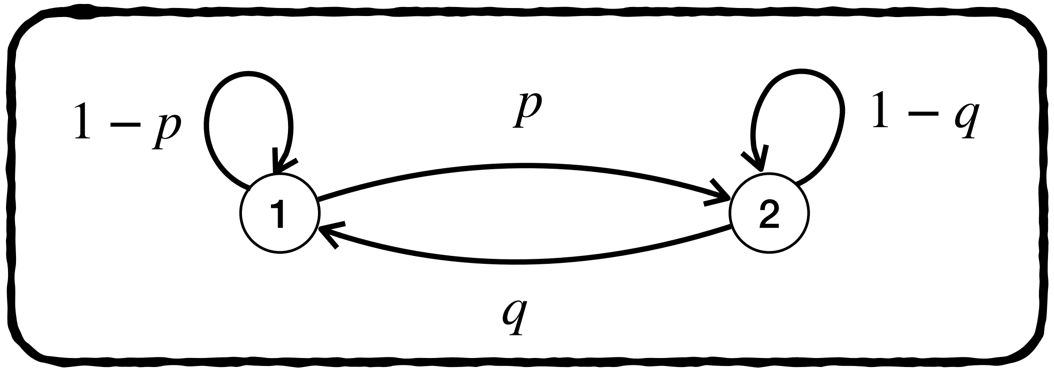

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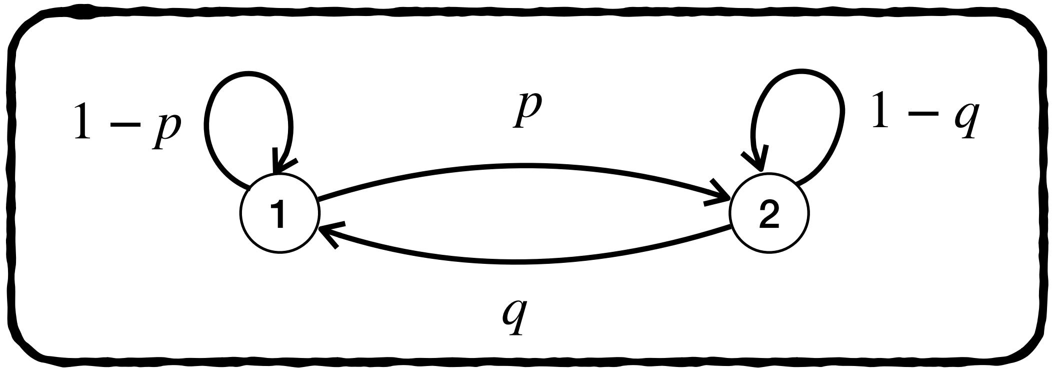

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Compute $\|\mu_0^T P^t - \pi^T\|$

$$\Delta_t = |\mu_t(1) - \pi(1)|$$

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$$\begin{aligned}
\Delta_{t+1} &= \left| \mu_{t+1}(1) - \frac{q}{p+q} \right| \\
&= \left| \mu_t(1-p) + (1-\mu_t(1))q - \frac{q}{p+q} \right| \\
&= (1-p-q)\left| \mu_t(1) - \frac{q}{p+q} \right| = (1-p-q) \cdot \Delta_t
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Since $p, q \in [0,1]$, there are two ways to prohibit $\Delta_t \to 0$: $p = q = 1$ or $p = q = 0$

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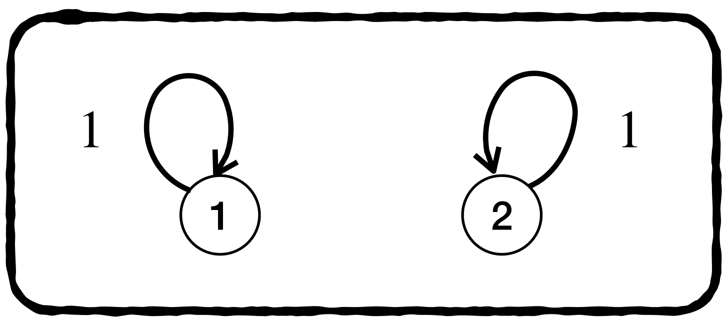

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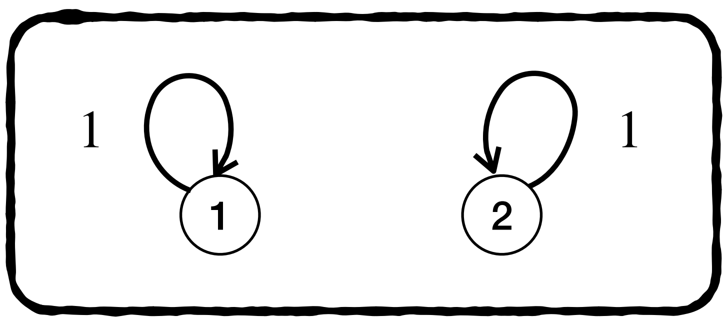

$$\forall t,\ \Delta_t = \Delta_0$$

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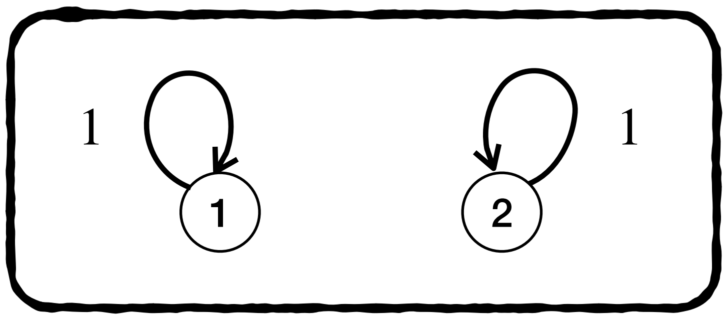

$$\forall t,\ \Delta_t = \Delta_0$$

The graph is disconnected

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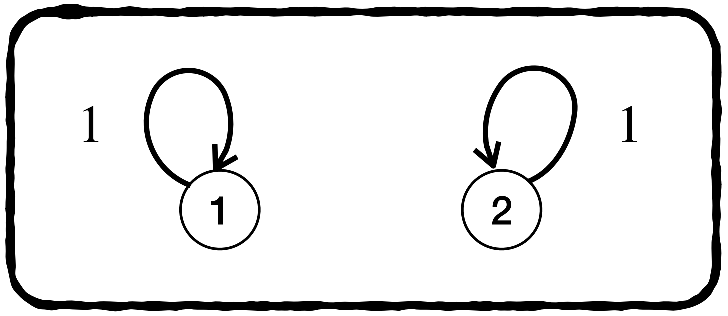

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The chain is called reducible

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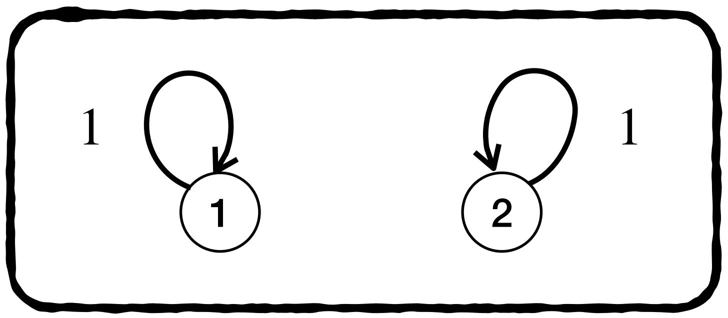

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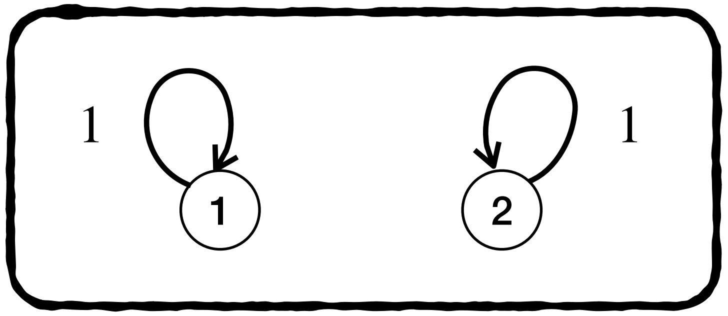

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Chain = convex combination of small chains

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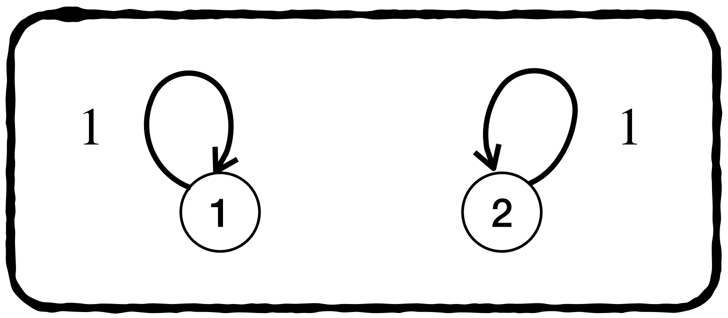

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Stationary distribution=convex combination of “small” distributions

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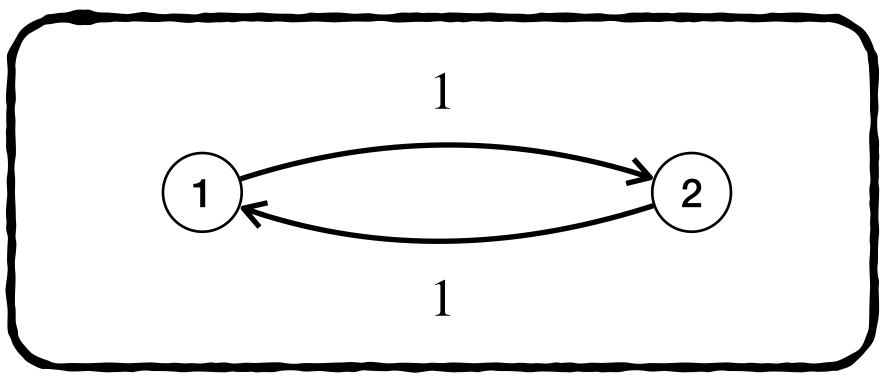

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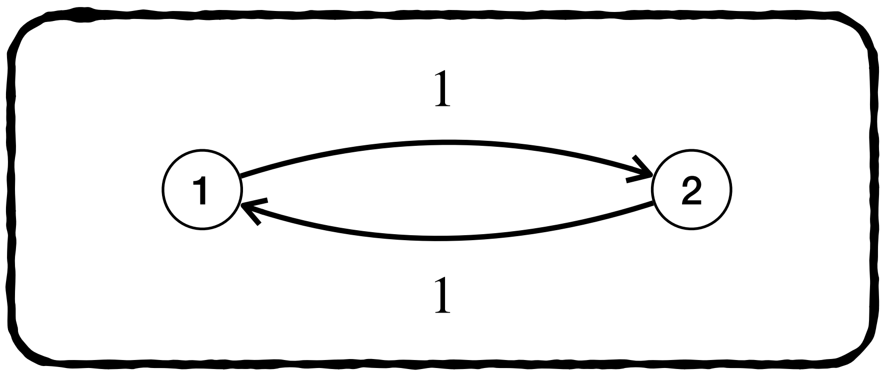

$$\forall t,\ \Delta_t = -\,\Delta_{t-1}$$

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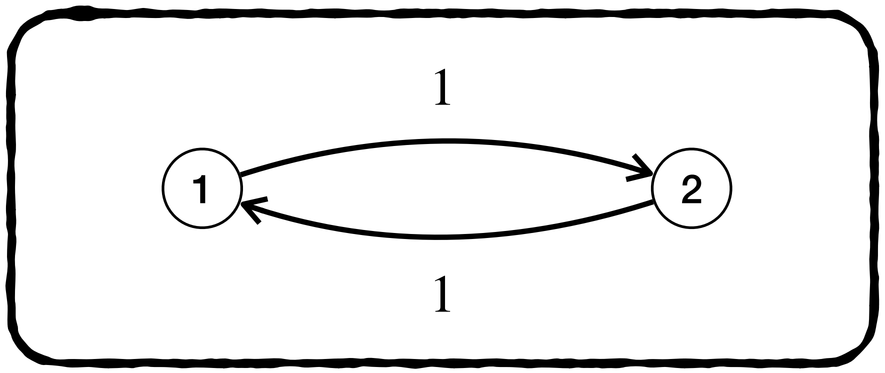

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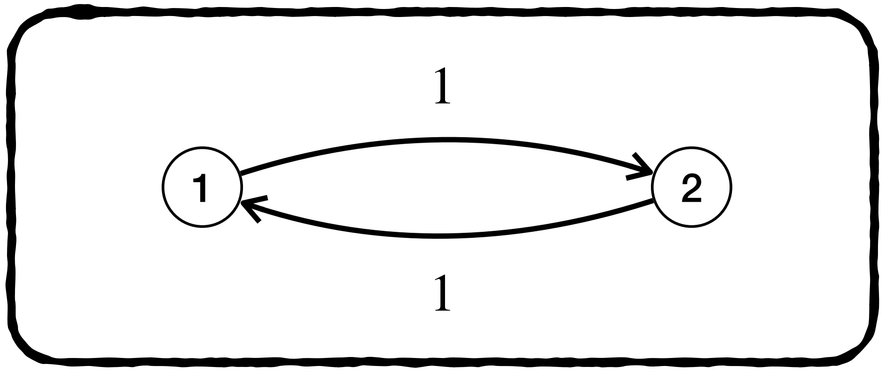

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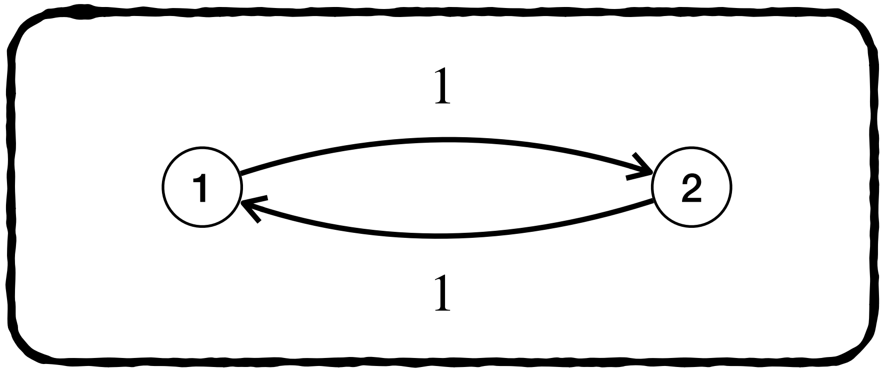

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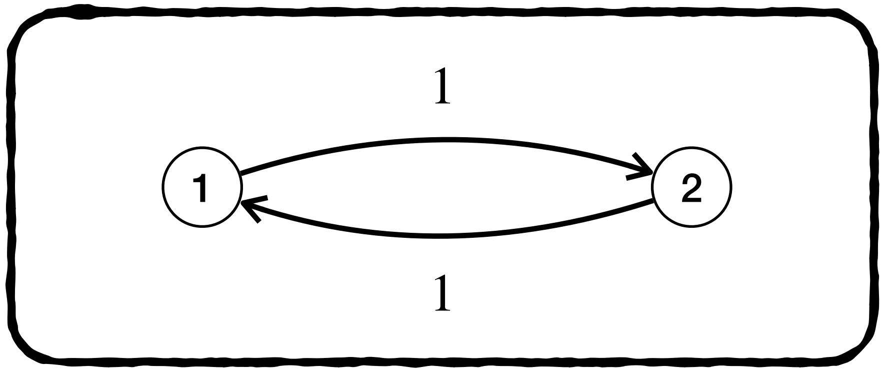

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In this case, not all initial distribution converges to the stationary distribution

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If a finite chain P is irreducible and aperiodic, then it has a unique stationary distribution π. Moreover, for any initial distribution μ, it holds that

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(Show on board, see the note for details)

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- We have powerful tools (spectral method) to analyze reversible chains

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An $n \times n$ symmetric matrix A has n real eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which are orthogonal. Moreover, it holds that

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Consider the Hilbert space $\mathbb{R}^n$ endowed with $\langle \cdot , \cdot \rangle_\pi$

Let $P \in \mathbb{R}^{n\times n}$ be reversible with respect to π. It has n real eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ which are orthogonal in $(\mathbb{R}^n, \langle \cdot\,, \rangle_\pi)$. Moreover

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Proof. Reduce to the symmetric case.

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Proof next week!

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If P is irreducible ($\lambda_{n-1} < 1$) and aperiodic ($\lambda_1 > -1$)

$$\lim_{t\to\infty} P^t = \mathbf{1}\mathbf{1}^T D_\pi = \begin{bmatrix} \pi^T \\ \pi^T \\ \vdots \\ \pi^T \end{bmatrix}$$