Advanced Algorithms IX (Fall 2020)

Instructor: Chihao Zhang
Scribed by: Jiabao Ji, Ruihang Lai

Last modified on Nov 17, 2020

In this lecture, we introduce discrete *stopping times* and *Optional Stopping Theorem* (OST). OST is a powerful tool for analyzing stochastic processes. We will present a few typical applications of OST.

1 Stopping Time

1.1 Review of Martingales

Recall the definition of martingales we introduced in the last lecture. Let $\{X_t\}_{t\geq 0}$ be a sequence of random variables, and $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration. We say $\{X_t\}_{t\geq 0}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ if

(i) X_t is $\mathcal{F}_t$-measurable, and

(ii) $\mathbf{E}[X_{t+1} \mid \mathcal{F}_t] = X_t$.

The following property of a martingale is immediate:

Lemma 1. *Let n be a fixed positive integer. If $\{X_t\}_{t\geq 0}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, then*

$$\mathbf{E}[X_n] = \mathbf{E}[X_0].$$

Proof. By taking expectation of the both sides of (ii), from the definition of conditional expectation we can derive that

$$\mathbf{E}[X_{t+1}] = \mathbf{E}[\mathbf{E}[X_{t+1} \mid \mathcal{F}_t]] = \mathbf{E}[X_t].$$

It follows that $\mathbf{E}[X_n] = \mathbf{E}[X_{n-1}] = \ldots = \mathbf{E}[X_0]$. □

Suppose that there is a gambler who participates in a sequence of fair gambling games, and the martingale $\{X_t\}_{t\geq 0}$ represents the winnings after each game. The above lemma states that, if the number of games played is initially fixed, then the expected total gain $\mathbf{E}[X_n]$ equals to $\mathbf{E}[X_0]$ which is 0.

Now consider the case that the number of games is not fixed at the beginning, i.e., the gambler choose to play a random number of games according to some strategies which may even involve the gain. This idea leads to the following definition of stopping time:

Definition 2. *Let $\tau \in \mathbb{N} \cup \{\infty\}$ be a random variable. We say τ is a stopping time if for all $t \geq 0$, the event "$\tau \leq t$" is $\mathcal{F}_t$-measurable.*

For example, the first time that the gambler wins five games in a row is a stopping time, since for a given t, this can be determined by looking at the outcomes of all the previous games, and therefore the time is $\mathcal{F}_t$-measurable. However, the *last* time the gambler wins five games in a row is *not* a stopping time, since determining whether the time is t cannot be done without knowing $X_{t+1}, X_{t+2}, \ldots$.

2 Optional Stopping Theorem

From Lemma 1 we know that $\mathbf{E}[X_n] = \mathbf{E}[X_0]$ for a fixed time n. It is natural to ask whether $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ for a given stopping time τ. This does not hold in general. To see this, we know that the time that the gambler first receives positive total gain is a stopping time. But the expected total gain when the gambler quits must be positive and thus cannot be $\mathbf{E}[X_0] = 0$. The Optional Stopping Theorem (OST) offers us some sufficient conditions for $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ to hold.

Theorem 3 (Optional Stopping Theorem). *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if <u>all</u> the three following conditions hold:*

(1) $\mathbf{Pr}[\tau < \infty] = 1$,

(2) $\mathbf{E}[|X_\tau|] < \infty$, *and*

(3) $\lim_{t\to\infty} \mathbf{E}\left[X_t \cdot \mathbb{1}_{[\tau > t]}\right] = 0$.

The conditions in Theorem 3 are sometimes not convenient to verify. In applications, we often use the following weaker version of OST:

Theorem 4. *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if <u>at least one of</u> the three following conditions holds:*

(1) τ *is bounded, or*

(2) $\mathbf{Pr}[\tau < \infty] = 1$, *and there is a finite M such that $|X_t| \leq M$ for all $t < \tau$, or*

(3) $\mathbf{E}[\tau] < \infty$, *and there is a constant c such that $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$ for all $t < \tau$.*

The proof of Theorem 3 and Theorem 4 can be found in standard textbooks on stochastic processes. In the following sections, we will introduce some applications.

3 1-D Random Walk, Part 1

Examples in this section justify the conditions in Theorem 4. Suppose there is a man walking randomly on a 1-D axis. Let $\{X_t\}_{t\geq 0}$ be the positions of the man at each time. Starting at $X_t = 0$, at time t, the man takes a step $c_t \in_R \{-1, 1\}$ and reach X_{t+1}, i.e., $X_{t+1} = X_t + c_t$. It is easy to verify that $\{X_t\}_{t\geq 0}$ is a martingale. In the following we define three different stopping times and check whether OST applies.

1. Let τ be the first time t such that $c_t = 1$. Then $\mathbf{E}[\tau] < \infty$ since by definition $\tau \sim \mathrm{Geom}\left(\frac{1}{2}\right)$, and $|X_{t+1} - X_t| \leq 1$ for all $t < \tau$. Therefore from the condition (3) of OST we have $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$. That is, if the man stops at the first time such that $c_\tau = 1$, then the expected final position is 0.

2. Let τ be the first time t such that $X_t = 1$. This process is called "1-d random walk with one absorbing barrier" and it is well-known that $\mathbf{E}[\tau] = \infty$. We can verify that no condition in Theorem 4 holds. Therefore, OST cannot be applied in this case.

3. Let τ be the minimum between 30 and the first time t such that $X_t = 1$. Note that we only slightly modify the definition of τ in case 2. But in this case, τ is at most 30, which satisfies condition (1). Therefore by OST $\mathbf{E}[X_\tau]$ is again 0.

4 1-D Random Walk, Part 2

In this part we step further to see the power of OST. The basic settings of the 1-D random walk here are the same as in section 3. Let $a, b > 0$ be two integers and τ be the time such that the man first reaches $-a$ or b, i.e., the first time t such that $X_t = -a$ or $X_t = b$. The model is called "1-d random walk with two absorbing barrier". The question here is: what is $\mathbf{E}[\tau]$?

We want to construct a martingale $\{Y_t\}_{t \geq 0}$ such that OST can be applied to $\{Y_t\}_{t \geq 0}$ and τ and thereby we can derive an equality related to $\mathbf{E}[\tau]$.

Before calculating $\mathbf{E}[\tau]$, we first determine the probability that the man stops at position $-a$, i.e., $\mathbf{Pr}[X_\tau = -a]$. Denote $P_a \triangleq \mathbf{Pr}[X_\tau = -a]$. Consider a time period $T = a + b$. In each period of time, if the man always walks along the positive direction, then he must have reached position b, since at the beginning of this period his position is between $[-a, b]$ and $-a + T = b$. The event that the man always walks along the positive direction in a period of time T happens with probability $2^{-(a+b)}$. Therefore the expected number of periods such that the man reaches position b is finite, similar to example 1 in section 3. Hence we have $\mathbf{E}[\tau] < \infty$. Then obviously τ and $\{X_t\}_{t \geq 0}$ satisfy the condition (3) of OST, and thus $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$. On the other hand, we have $\mathbf{E}[X_\tau] = P_a \cdot (-a) + (1 - P_a) \cdot b$. This gives $P_a = \frac{b}{a+b}$.

Then for all $t \geq 0$ we define a random variable $Y_t \triangleq X_t^2 - t$ which involves time[1]. We have the following claim:

Claim 5. $\{Y_t\}_{t \geq 0}$ *is a martingale.*

Proof. We verify the claim by definition. First we have

$$\begin{aligned}
\mathbf{E}[Y_{t+1} \mid \mathcal{F}_t] &= \mathbf{E}\left[X_{t+1}^2 - (t+1) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbf{E}\left[(X_t + c_t)^2 - (t+1) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbf{E}\left[X_t^2 \,\middle|\, \mathcal{F}_t\right] + 2\mathbf{E}[X_t c_t \mid \mathcal{F}_t] + \mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] - (t+1).
\end{aligned}$$

Since X_t is $\mathcal{F}_t$-measurable, $\mathbf{E}[c_t \mid \mathcal{F}_t] = 0$ and $\mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] = 1$, we can further derive that

$$\mathbf{E}[Y_{t+1} \mid \mathcal{F}_t] = X_t^2 + 0 + 1 - (t+1) = X_t^2 - t = Y_t,$$

hence $\{Y_t\}_{t \geq 0}$ is a martingale. □

Note that $X_t \in [-a, b]$ for all $t \geq 0$. Thus $|Y_{t+1} - Y_t| = \left|X_{t+1}^2 - (t+1) - X_t^2 + t\right| = \left|X_{t+1}^2 - X_t^2 - 1\right|$ is bounded by some constant c. And we have proved that $\mathbf{E}[\tau]$ is finite. Therefore by the condition (3) we have $\mathbf{E}[Y_\tau] = \mathbf{E}[Y_0] = 0$. On the other hand we have $\mathbf{E}[Y_\tau] = \mathbf{E}\left[X_\tau^2\right] - \mathbf{E}[\tau]$ by definition, and thus

$$\mathbf{E}[\tau] = \mathbf{E}\left[X_\tau^2\right] = a^2 P_a + b^2 (1 - P_a) = a^2 \cdot \frac{b}{a+b} + b^2 \cdot \frac{a}{a+b} = ab.$$

This means that the expected time until the man stops is ab.

5 Bertrand's Ballot Theorem

We introduce another application of OST: Bertrand's Ballot Theorem. Although the problem can be solved via counting argument, we provide an elegant martingale based proof. The exposition here is from [2].

[1] The construction of Y is a useful trick and generalizes to arbitrary martingales

Suppose we have two candidates Alice and Bob running for an election, with Alice obtaining a votes in total and Bob obtaining b votes where $b < a$. We count these votes in a random order, chosen uniformly at random from all permutations on $a + b$ votes. We have the following theorem:

Theorem 6 (Bertrand's Ballot Theorem). *The probability that Alice is always ahead in the count is* $\frac{a-b}{a+b}$.

Let $n = a + b$ be the total number of votes, and S_k be the number of votes by which candidate Alice is leading after k votes are counted. Then we have $S_n = a - b$. For all $0 \leq k \leq n - 1$, we define

$$X_k \triangleq \frac{S_{n-k}}{n-k}.$$

Note that X_k can be negative.

Claim 7. $\{X_k\}_{0 \leq k \leq n-1}$ *is a martingale with respect to the filtration* $\{\sigma(X_0, \ldots, X_k)\}_{0 \leq k \leq n-1}$.

Proof. Note that $X_0, X_1, \ldots, X_{n-1}$ relates to the counting process in a backward order, that is, X_0 is related to S_n, X_{k-1} is related to S_{n-k+1}. Therefore, for $\mathbf{E}[X_k \mid X_0, \ldots, X_{k-1}]$, X_k is conditioned on $X_0, \ldots, X_{k-1}$, which is equivalent to be conditioned on $S_n, S_{n-1}, \ldots, S_{n-k+1}$.

Conditioning on S_{n-k+1}, since Alice is leading by S_{n-k+1} votes after counting $n - k + 1$ votes, we can solve that

$$\text{Alice has } \frac{n-k+1+S_{n-k+1}}{2} \text{ votes, and}$$
$$\text{Bob has } \frac{n-k+1-S_{n-k+1}}{2} \text{ votes.}$$

The $(n - k + 1)$-th vote in the count is a random vote from among these first $n-k+1$ votes. Also, S_{n-k} is equal to $S_{n-k+1} + 1$ if the $(n - k + 1)$-th vote was for Bob, and equal to $S_{n-k+1} - 1$ if that vote was for Alice. Thus, for all $k \geq 1$, we have

$$\begin{aligned}
\mathbf{E}[S_{n-k} \mid S_{n-k+1}] &= (S_{n-k+1} + 1) \cdot \frac{n-k+1-S_{n-k+1}}{2(n-k+1)} \\
&\quad + (S_{n-k+1} - 1) \cdot \frac{n-k+1+S_{n-k+1}}{2(n-k+1)} \\
&= S_{n-k+1} \cdot \frac{n-k}{n-k+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}[X_k \mid X_0, \ldots, X_{k-1}] &= \mathbf{E}\left[\frac{S_{n-k}}{n-k} \,\middle|\, S_n, \ldots, S_{n-k+1}\right] \\
&= \frac{S_{n-k+1}}{n-k+1} \\
&= X_{k-1},
\end{aligned}$$

which means that $\{X_0, X_1, \ldots, X_{n-1}\}$ is a martingale. □

Define the stopping time $\tau \triangleq \min\{k \mid X_k = 0\} \wedge (n-1)$. Obviously, τ is a bounded stopping time, satisfying the condition (1) of OST. Therefore,

$$\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = \frac{\mathbf{E}[S_n]}{n} = \frac{a-b}{a+b}.$$

Now we consider the following two events:

E1: Alice leads throughout the count. Then all S_{n-k} and X_k are positive for $0 \le k \le n-1, \tau = n-1$, thus

$$X_\tau = X_{n-1} = S_1 = 1,$$

where $S_1 = 1$ follows from the fact that Alice must receive the first vote in the count.

E2: Alice does not lead throughout the count. There must be some $k < n-1$ such that $X_k = 0$ since we know Alice receives more votes in the end and once Bob leads, there must be some intermediate point k where $S_k = 0$, leading to $X_k = 0$. In this case, $\tau = k < n-1$ and $X_\tau = 0$.

Therefore, we have

$$\mathbf{E}\left[X_\tau\right] = \frac{a-b}{a+b} = 1 \cdot \mathbf{Pr}\left[\mathbf{E1}\right] + 0 \cdot \mathbf{Pr}\left[\mathbf{E2}\right].$$

Thus we have

$$\mathbf{Pr}\left[\mathbf{E1}\right] = \mathbf{Pr}\left[\text{Alice leads throughout the count}\right] = \frac{a-b}{a+b}.$$

6 Waiting Time for Patterns

Recall an exercise in homework 1:

> **Problem 2.** (*Probability and Computing* **2.24**)
> We roll a standard fair die over and over. What is the expected number of rolls until the first pair of consecutive sixes appears? (Hint: The answer is not 36.)

In the above exercise, we are asked to find out the expected number of rolls until the first "66" appears. In this section, we study this family of "pattern matching" problems using martingale approach. The brilliant idea originally appeared in [1]. Suppose we flip a fair coin, with 1 denoting "head" and 0 denoting "tail". Given a pattern $p \in \{0,1\}^*$, we want to ask how many flips do we need to see the pattern p in expectation.

Suppose the pattern is $p = p_1 p_2 \dots p_k$ and the result sequence of flipping is $B = b_1 b_2 \dots$. We now consider a gamble with infinite gamblers playing. For all $j \ge 1$, the strategy for gambler j is as following:

(i) At time j, he bets ¥1 on "$b_j = p_1$".

(ii) If he wins at time j, he bets ¥2 on "$b_{j+1} = p_2$" at time $j+1$.

(iii) He keeps doubling the stakes until he loses.

Note that j borrows ¥1 at beginning, and if he loses the gamble at some time, he only loses this borrowed ¥1.

Let $G_j(t)$ be the money of gambler j at time t. It follows that $\left\{G_j(t)\right\}_{t\ge 0}$ is a martingale. This is because gambler j is playing a fair gamble game with 50% chance of win and 50% chance of lost, and thus the expected gains remain unchanged. Now we define $X(t) \triangleq \sum_j G_j(t)$, which is the money of all gamblers at time t. Note that for a fixed time t, $G_j(t)$ is always 0 when $j > t$ since those gamblers have not started their games yet. It means that $X(t)$ is actually a finite sum of $G_j(t)$s, and therefore $\{X(t)\}_{t\ge 0}$ is also a martingale.

Now we try to calculate the expectation in a similar way to the proof of theorem 6. Define τ as the first time that pattern p appears. τ and $\{X(t)\}_{t\ge 0}$ satisfy condition (3) of OST. Therefore with OST, we have $\mathbf{E}\left[X(\tau)\right] = \mathbf{E}\left[X(0)\right] = 0$. Then consider $\mathbf{E}\left[X(\tau)\right]$ in another way. There are three kinds of gamblers:

1. The first $\tau - k$ gamblers lose ¥1 each.
2. $G_{\tau-k+1}$ wins ¥$2^k - 1$ since this is the only gambler that wins all k games.
3. $G_{\tau-j+1}$ wins ¥$2^j - 1$ if and only if $p_1 \dots p_j = p_{k-j+1} \dots p_k$ since they are still playing when p first appears.

Let $\chi_j \triangleq \mathbb{1}_{[p_1 \dots p_j = p_{k-j+1} \dots p_k]}$, we have

$$X(\tau) = -\left(\tau - \sum_{j=1}^{k} \chi_j\right) + \sum_{j=1}^{k} \chi_j \left(2^j - 1\right).$$

Since $\mathbf{E}[X(\tau)] = 0$, by taking expectation over the above equation we can derive that

$$\mathbf{E}[X(\tau)] = -\mathbf{E}[\tau] + \sum_{j=1}^{k} \chi_j \cdot 2^j = 0.$$

Hence

$$\mathbf{E}[\tau] = \sum_{j=1}^{k} \chi_j \cdot 2^j. \tag{1}$$

As a sanity test, let's calculate a concrete pattern $p = 00$ using 1. We have $\mathbf{E}[\tau] = \sum_{j=1}^{2} \chi_j 2^j = 1 \cdot 2 + 1 \cdot 4 = 6$. We can also directly compute the value using definitions. Denoting $\mathbf{E}[\tau] = a$, and we have

$$a = \frac{1}{2}(1+a) + \frac{1}{2}\left[\frac{1}{2} \cdot 2 + \frac{1}{2}(2+a)\right].$$

This also gives $a = 6$.

References

[1] S.-Y. R. Li, *A martingale approach to the study of occurrence of sequence patterns in repeated experiments*, the Annals of Probability, (1980), pp. 1171–1176. 5

[2] M. Mitzenmacher and E. Upfal, *Probability and computing: Randomization and probabilistic techniques in algorithms and data analysis*, Cambridge university press, 2017. 3